



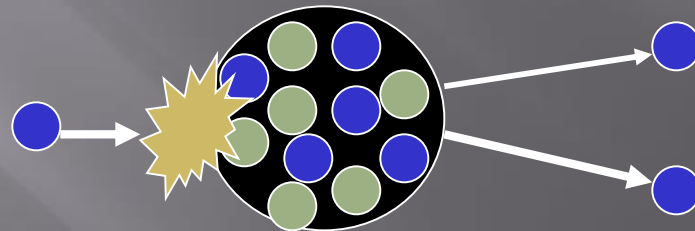
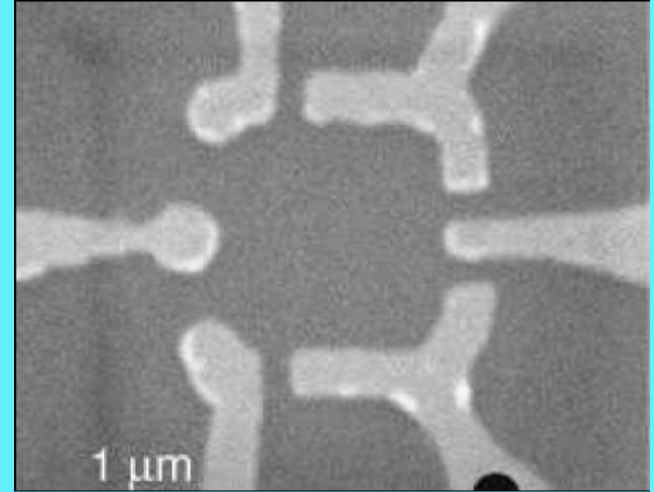
The Abdus Salam  
**International Centre  
for Theoretical Physics**

# RANDOM MATRIX THEORY AND ANDERSON LOCALIZATION I: ERGODIC AND NON-ERGODIC EXTENDED STATES IN RMT

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# Water waves, complex nuclei, single-electron devices (quantum dots)



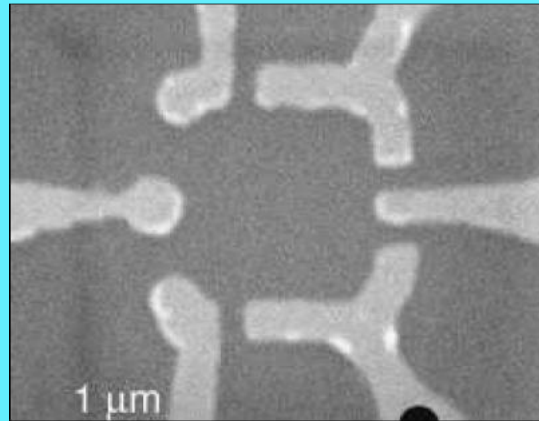
What is in common?

# Wave nature of electrons: Quantum dot is a chaotic billiard

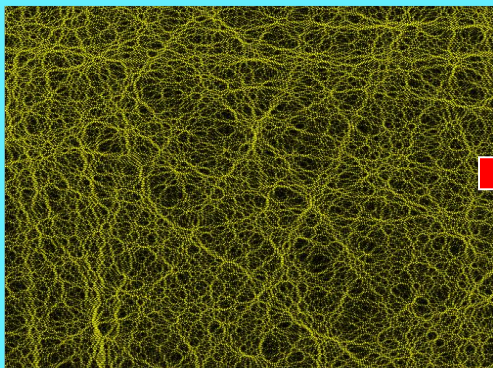
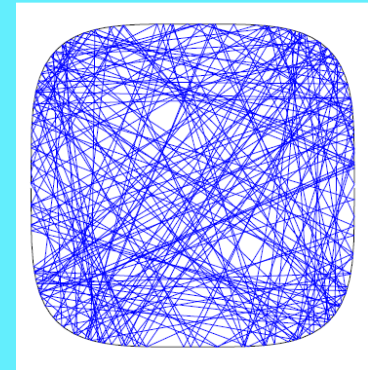
**Sea waves**



**Quantum dot**



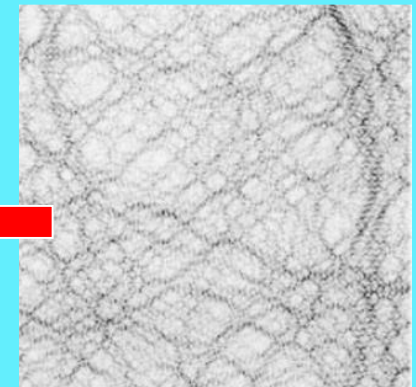
**Chaotic billiard**



**Random  
superposition of  
plane waves**

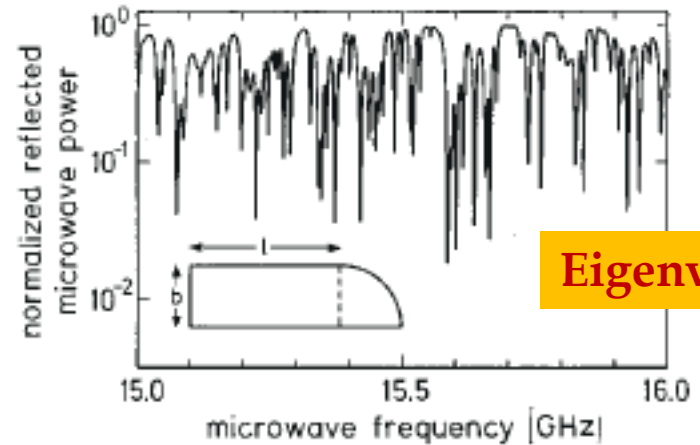
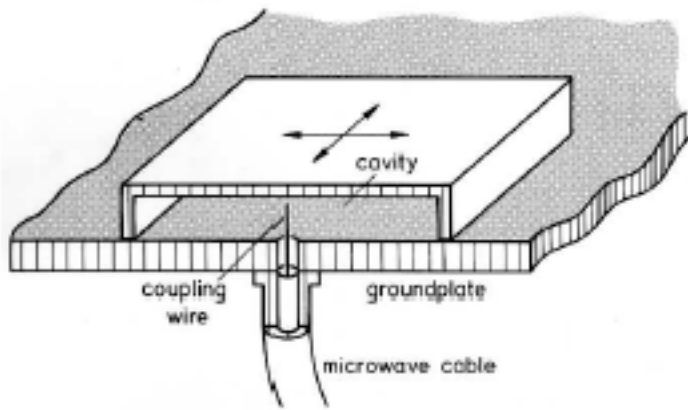


$$\Psi(r) = \sum a_{\vec{k}} \exp(i\vec{k}r)$$



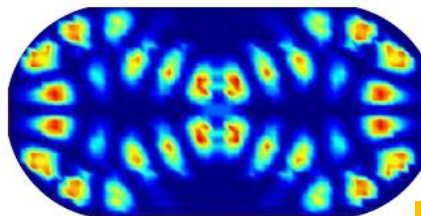
Wavefunction amplitude in  
a chaotic billiard

# Microwave billiards



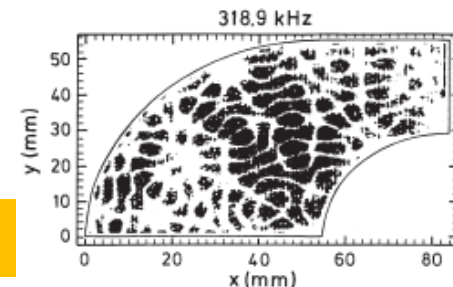
Eigenvalues

FIGURE 3 – Microwave set-up to study spectra and wave functions (left), and a typical microwave reflection spectrum (right) [2].



Eigenfunctions

FIGURE 4 – Wave functions in a stadium-shaped microwave resonator. The plot shows  $|\psi_n(r)|^2$  in a colour plot.



# Spectrum of complex nuclei

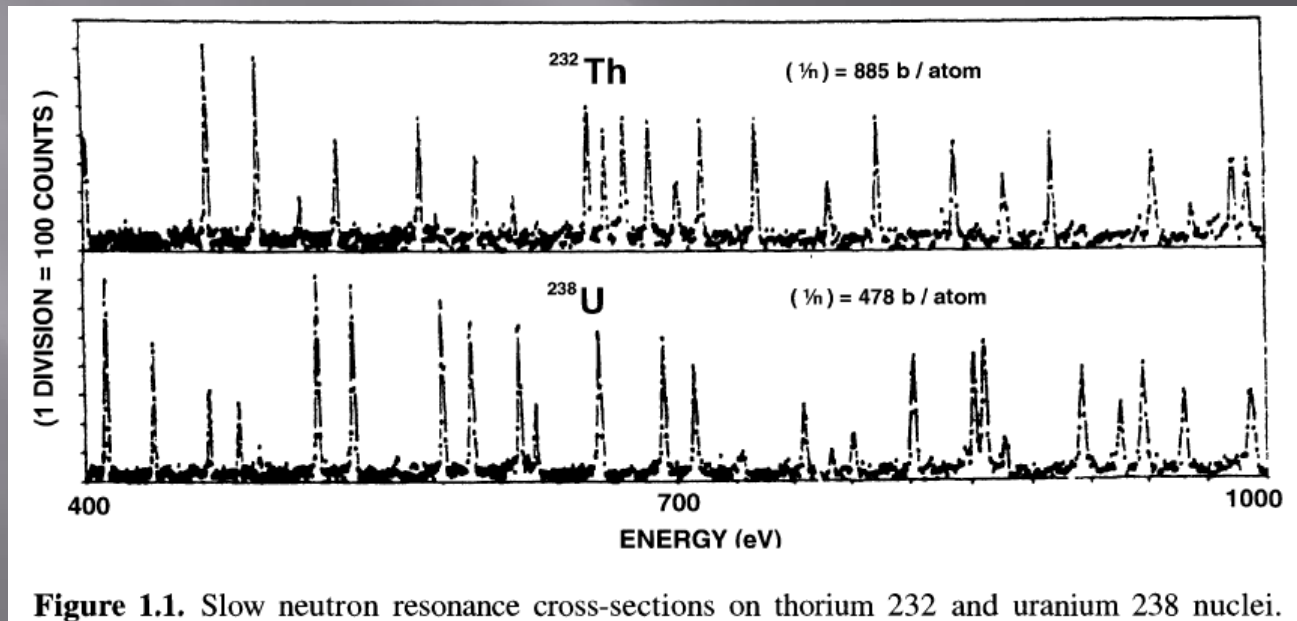
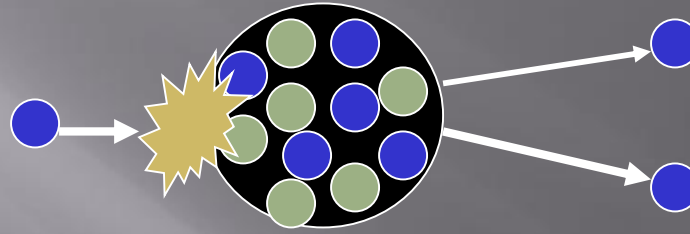
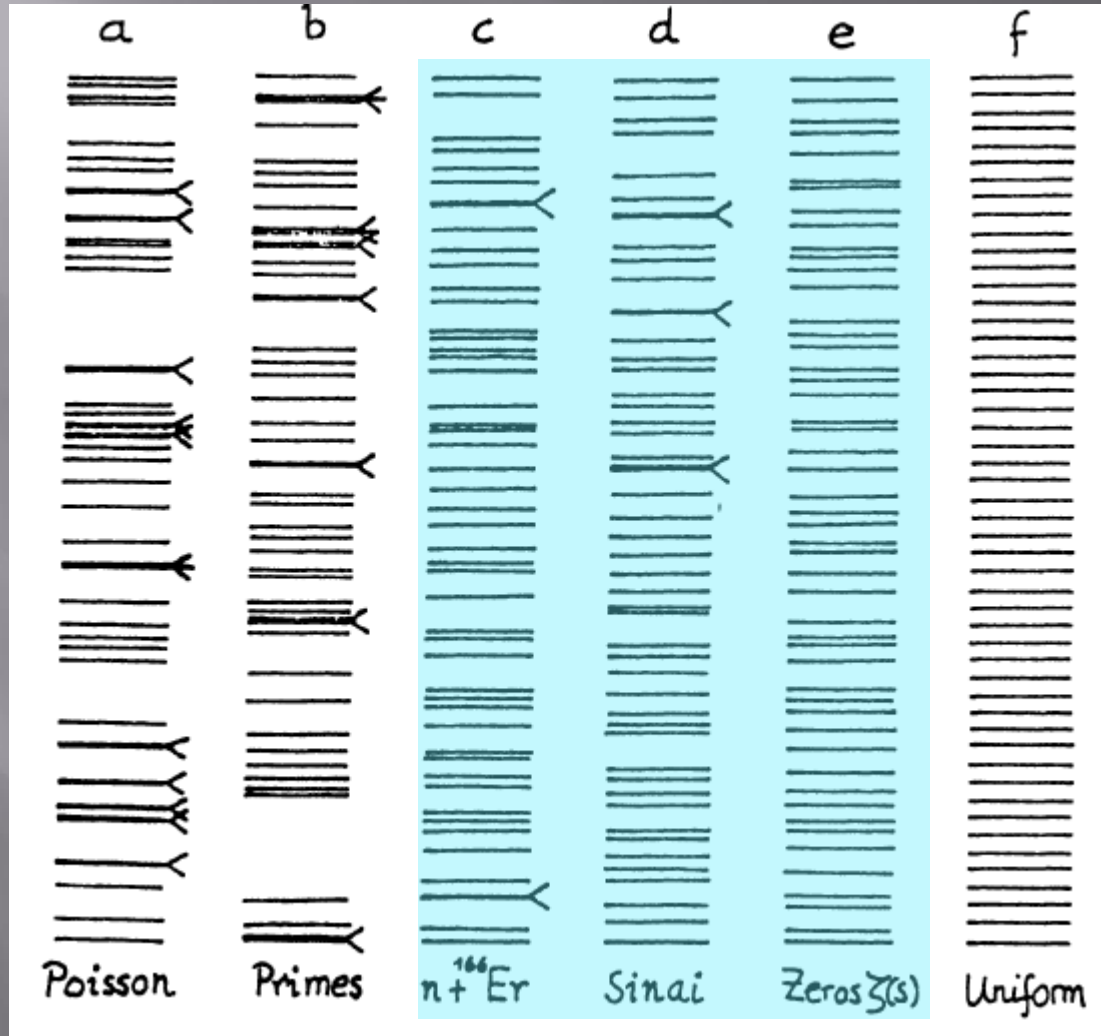
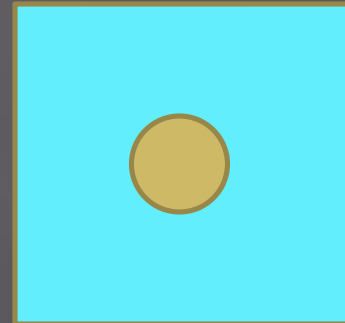


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei.

# The spectra



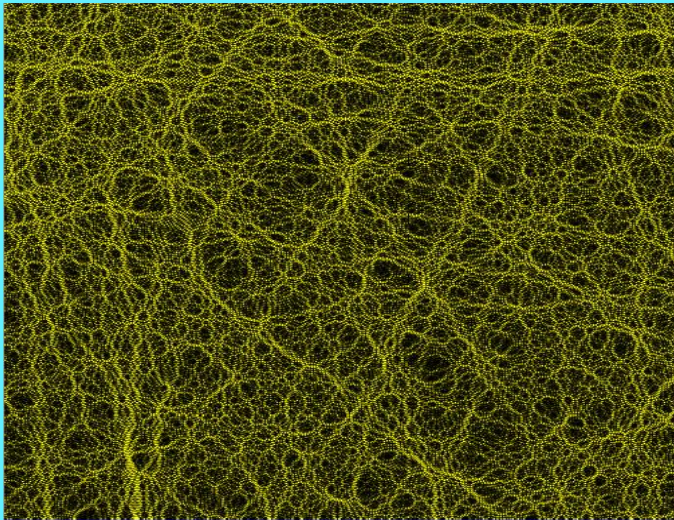
Sinai billiard



$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$z = \frac{1}{2} + iE$$

# Random plane waves



Random superposition of plane waves on an energy shell  $E=\text{const}$

$$\Psi(\mathbf{r}) = \sum_{\vec{k}} a_{\vec{k}} \exp(i\vec{k}\cdot\vec{r})$$

Rotation of basis:

$$\exp(i\vec{k}\cdot\vec{r}) \Rightarrow \sum_{\vec{q}} u_{\vec{k},\vec{q}} \exp(i\vec{q}\cdot\vec{r})$$

Does not change statistics of wavefunction (**basis invariance**)

Large number of terms in the sum over  $\mathbf{k}$  (large energy  $E$ )

CLT  $\Rightarrow$  Gaussian statistics

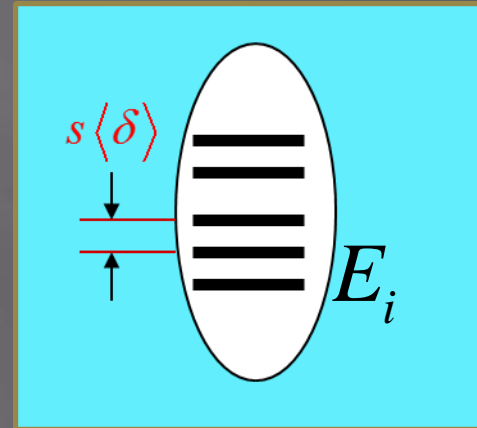
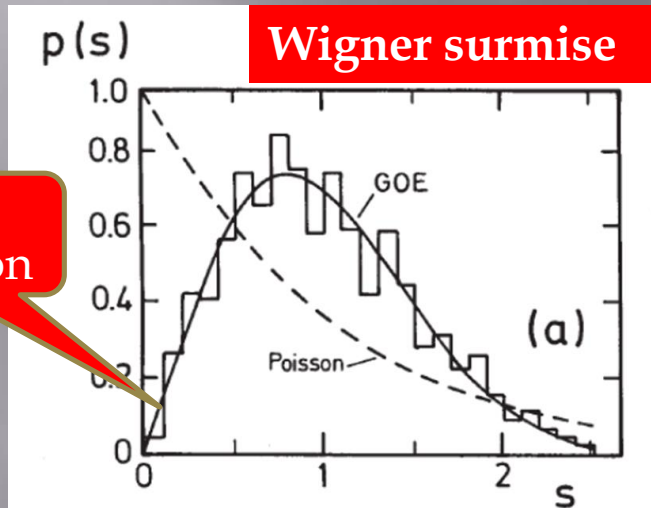
# Gaussian ensembles

$$H = \begin{pmatrix} 2.3 & -1.5 & -0.9 & 1.4 \\ -1.5 & -0.4 & 1.6 & 2.1 \\ -0.9 & 1.6 & -0.7 & 0.1 \\ 1.4 & 2.1 & 0.1 & 1.3 \end{pmatrix}$$

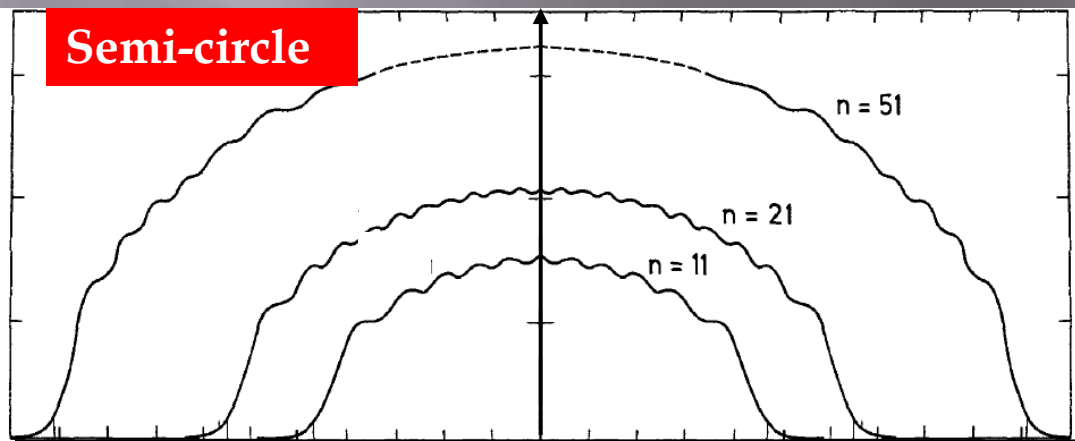
Random entry  $H_{nm}$ :  
fluctuates around zero  
independently of other  
entries and with  
Gaussian distribution

$$P(H) = \prod_{nm} e^{-H_{nm}^2} = e^{-\text{Tr}(H^2)}$$

# Famous features of Gaussian ensembles



**Semi-circle**



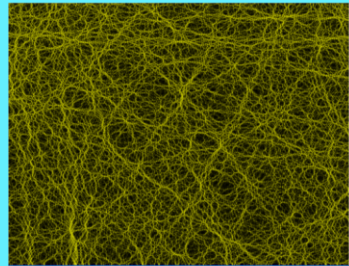
$$\rho_0(E) = N^{-1} \left\langle \sum_i \delta(E - E_i) \right\rangle$$

# Basis-rotation invariance

$$H \Rightarrow UHU^{-1}$$



Mathematically  
very powerful but  
excludes  
localization



Random superposition of plane waves on an  
energy shell  $E=\text{const}$

$$\Psi(r) = \sum_{\vec{k}} a_{\vec{k}} \exp(i\vec{k}r)$$

Rotation of basis:

$$\exp(i\vec{k}r) \Rightarrow \sum_q u_{k,q} \exp(i\vec{q}r)$$

Does not change statistics of  
wavefunction (**basis invariance**)

$$P(H) = e^{-\text{Tr}H^2} = e^{\text{Tr}(UHU^{-1})^2}$$

Does not change  
probability

# HOW TO GET LOCALIZATION IN RMT?

Basis Preference

# Rosenzweig-Porter RMT

Scaling with  
matrix size  
as for YS

$$\sigma = \frac{\lambda^2}{N^\gamma} \ll 1$$

$N \times N$  matrix,  
**uncorrelated**  
random entries

$$\langle H_{nm} \rangle = 0$$

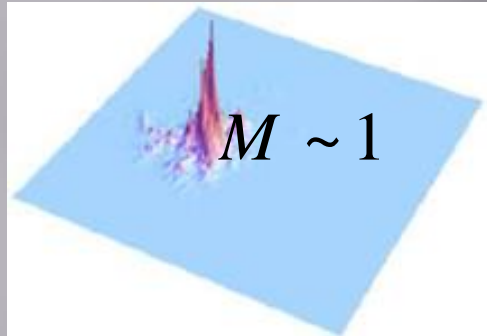
Special diagonal:  
Rosenzweig-  
Porter (1960)  
ensemble

Basis  
preference:  
Localization  
may happen  
in this  
particular  
fixed basis

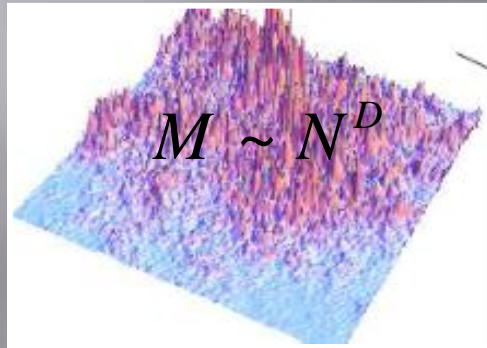
$$\langle H_{nm}^2 \rangle = \begin{pmatrix} 1 & \sigma & \sigma & \sigma \\ \sigma & 1 & \sigma & \sigma \\ \sigma & \sigma & 1 & \sigma \\ \sigma & \sigma & \sigma & 1 \end{pmatrix}$$

$$P(H) = \prod_{n \neq m} e^{-H_{nm}^2/2\sigma} \prod_n e^{-H_{nn}^2/2} \neq e^{-a \text{Tr}(H^2)}$$

# LOCALIZED, EXTENDED ERGODIC AND EXTENDED NON-ERGODIC PHASES

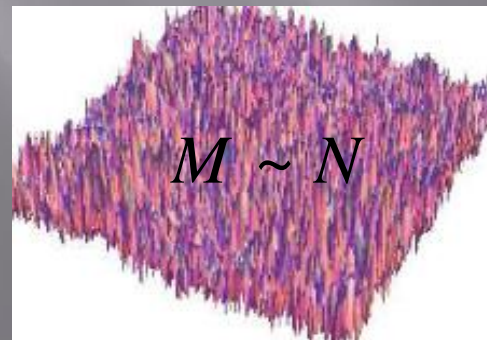


Finite number of occupied sites  
in thermodynamic limit **D=0**



Infinite number of occupied sites  
but zero fraction of all sites  
in the thermodynamic limit

**$0 < D < 1$**



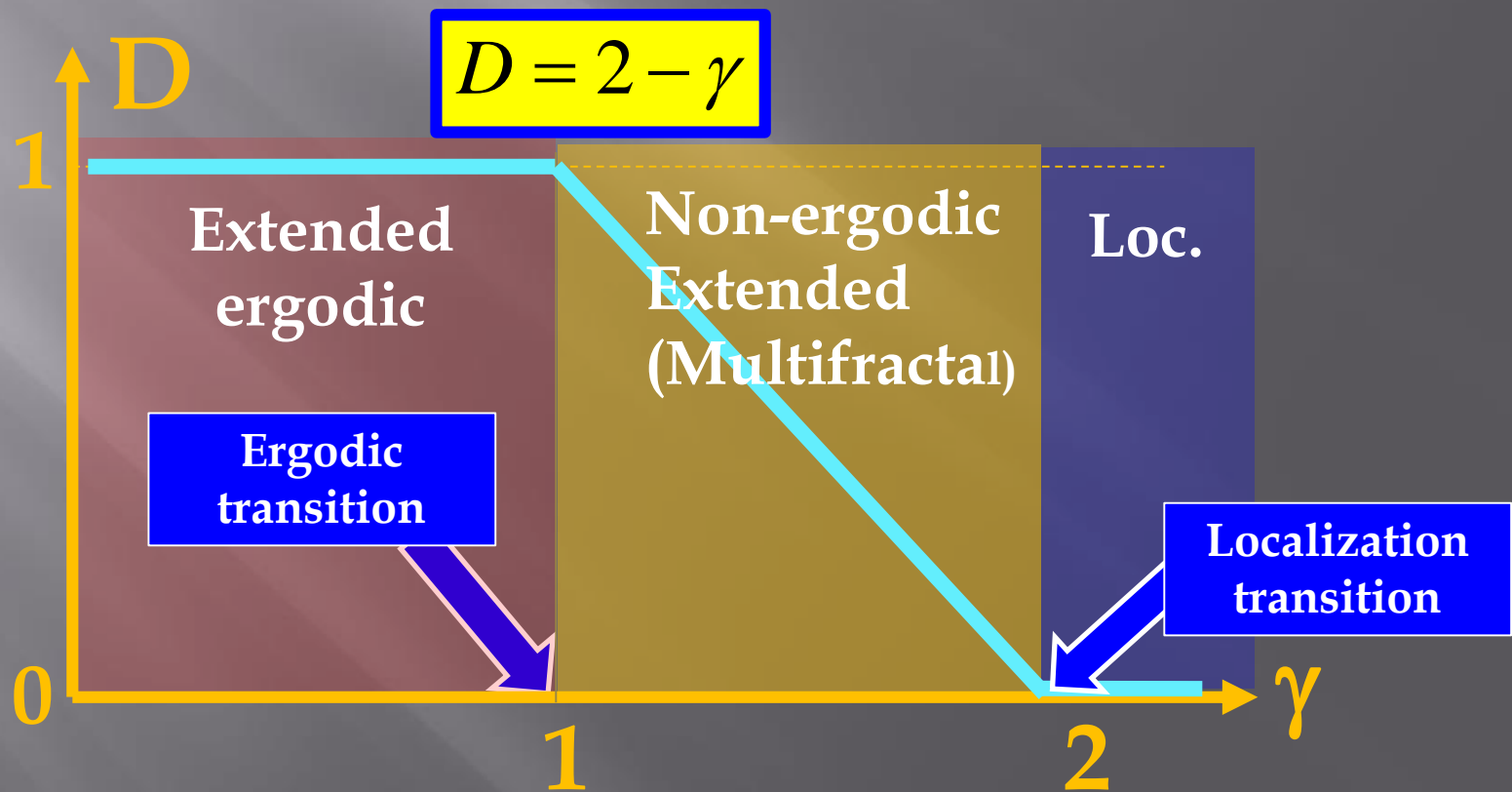
Finite fraction of occupied sites

**D=1**

# Non-ergodic extended phase and ergodic transition in RP RMT

V.E.K., I.M. Khaymovich,  
E. Cuevas, M. Amini,  
New J. Phys., v.17, 12202  
(2015)

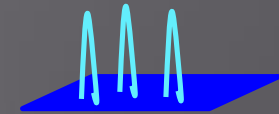
$$\langle |H_{nm}|^2 \rangle = \begin{cases} 1, & \text{if } n = m \\ \frac{\lambda^2}{N^\gamma}, & \text{if } n \neq m \end{cases}$$



# Number of sites in resonance

$$(\Delta\varepsilon)^2 = (H_{nn} - H_{mm})^2 < |H_{nm}|^2$$

$$P_{res} = N^{-\gamma}$$



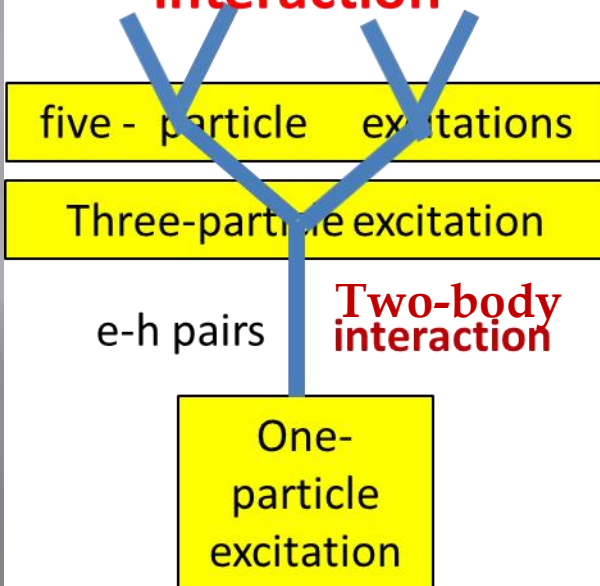
$$\#resonances = \#pairs \times probability = N^2 N^{-\gamma}$$

Competition of power laws

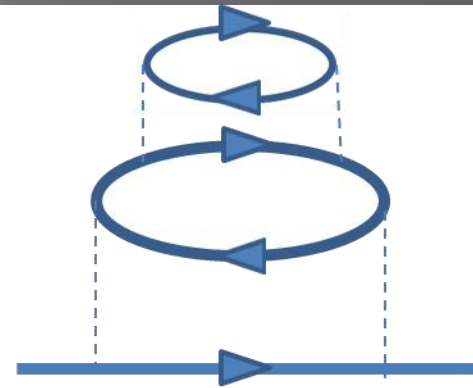
$$D = 2 - \gamma$$

# The structure of Fock/Hilbert space of interacting systems

**Tree-like structure  
of many-body  
interaction**



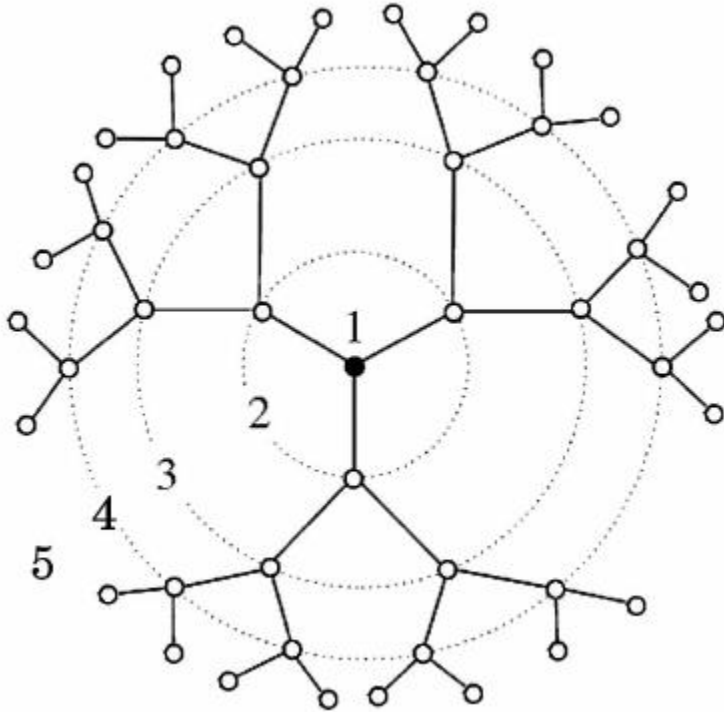
Altshuler, Gefen, Kamenev,  
Levitov, 1997



Basko, Aleiner, Altshuler,  
2005



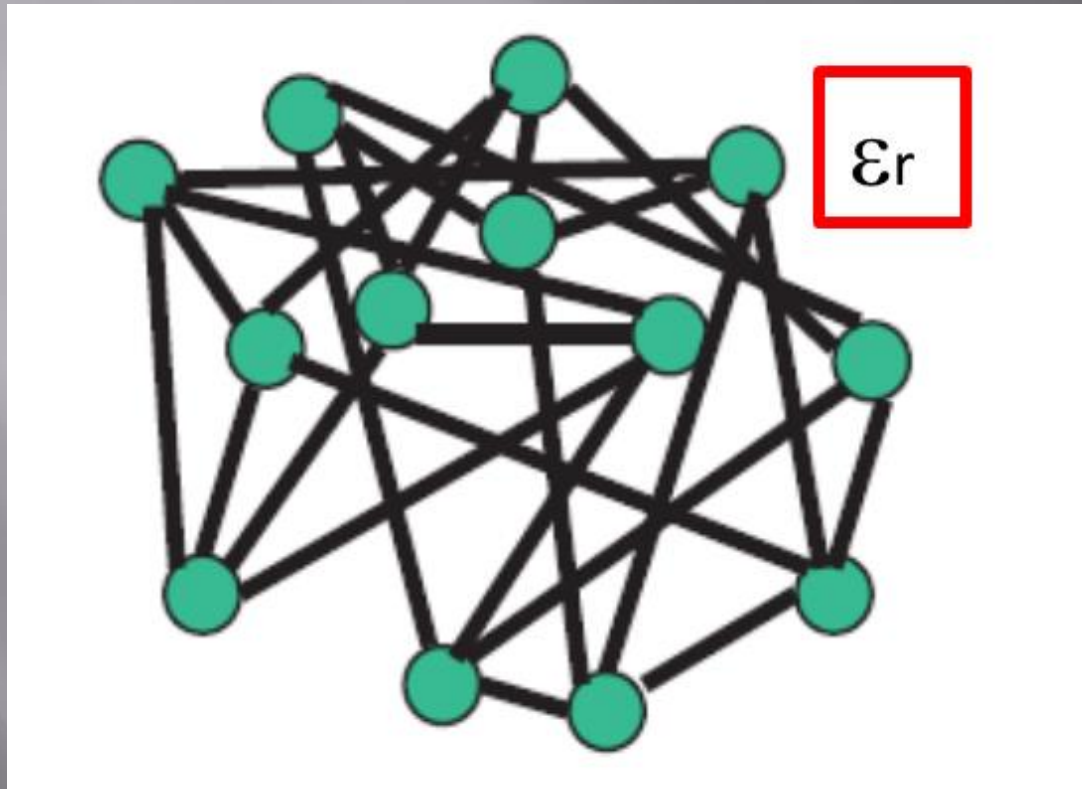
# Cayley tree



**No loops like in 1D  
but  $N(r)=K^r$  like in  
infinite dimensions**

**Coordination  
number finite like in  
finite dimensions**

...but no boundary

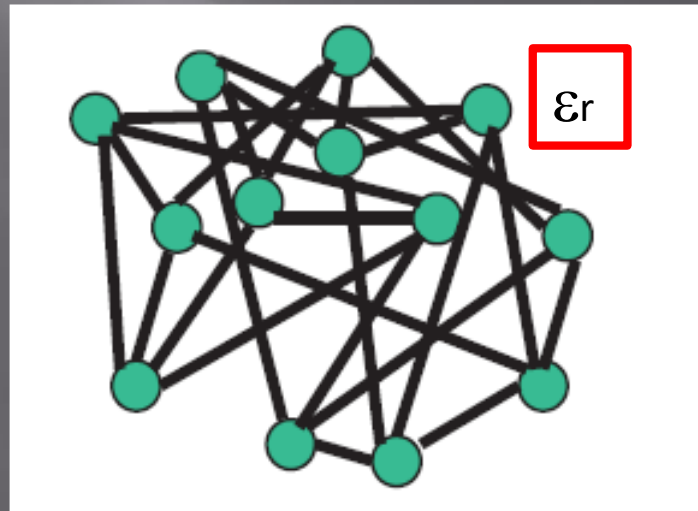
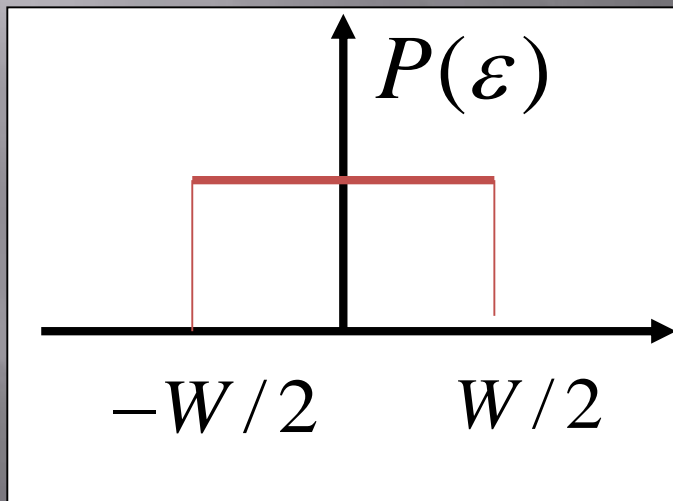


Random  
regular  
graph  
(RRG)

# Anderson model of localization

$$H = -I \sum_{\langle r, r' \rangle} c_r^+ c_{r'} + \sum_r \epsilon_r n_r$$

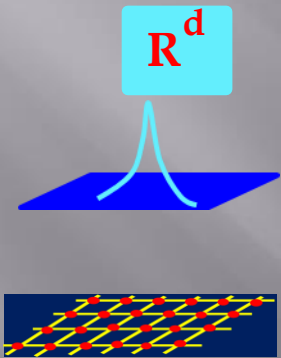
Disorder strength  $W$



# The case of RRG

$$P_{res}(r) = \exp[-\lambda r]$$

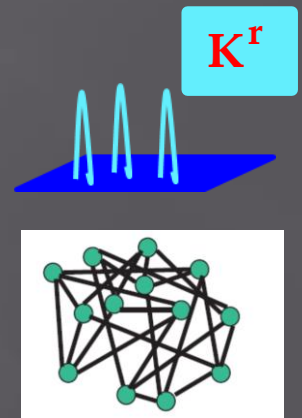
# pairs at a distance  $r$ :  $K^r = \exp[\ln K r]$



**Competition of exponentials**

$$\# \text{ res} = \exp[(\ln K - \lambda(W))d] = N^{(1-\lambda/\ln K)}$$

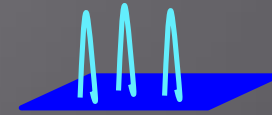
$d = \ln N / \ln K$  is the graph diameter



$$D = 1 - \lambda(W) / \ln K$$

# Number of sites in resonance

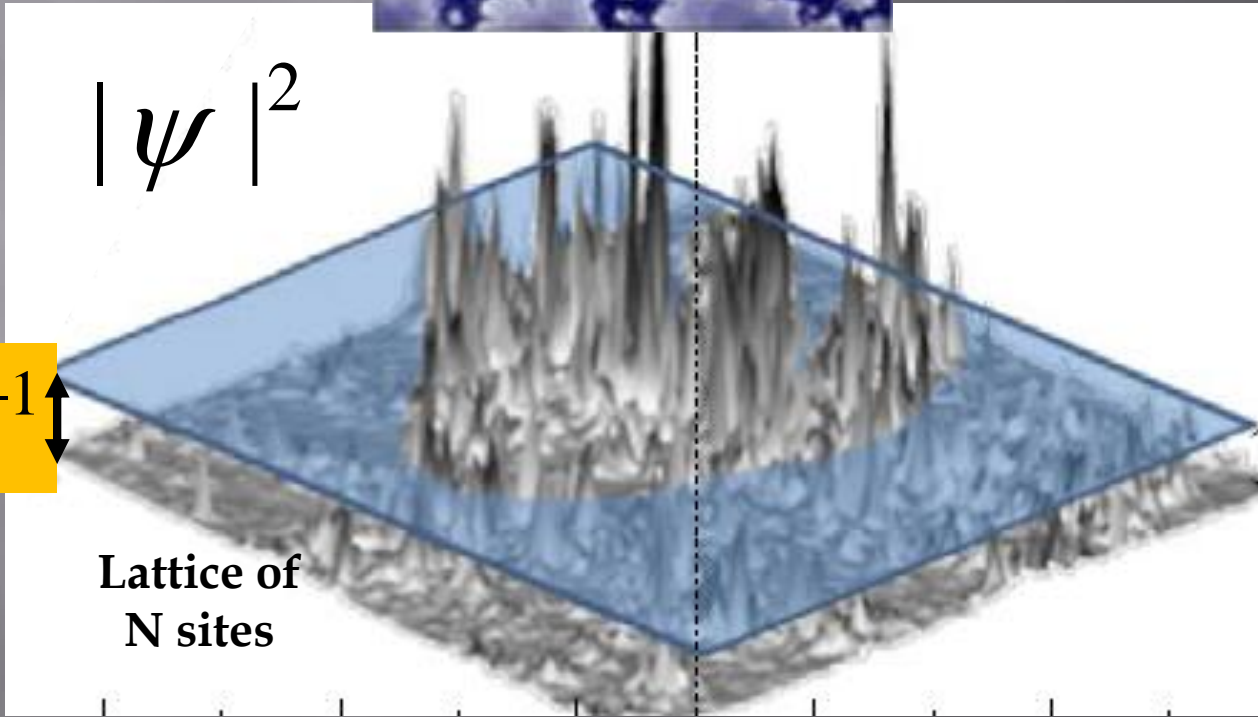
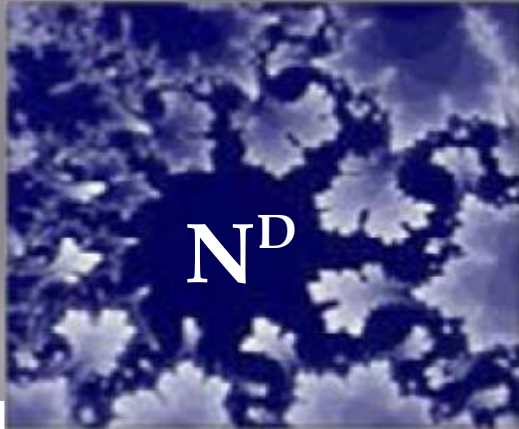
$$P_{res} = N^{-\gamma}$$



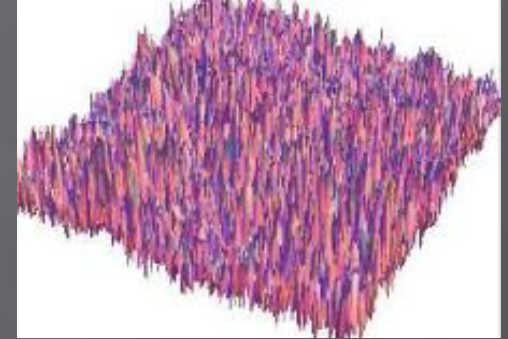
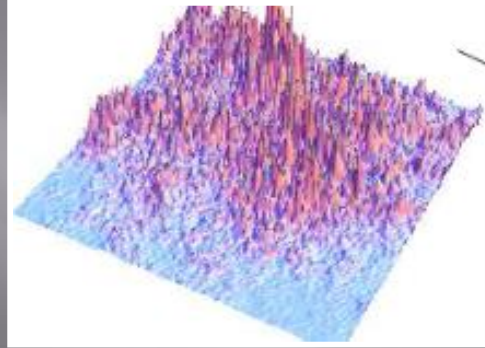
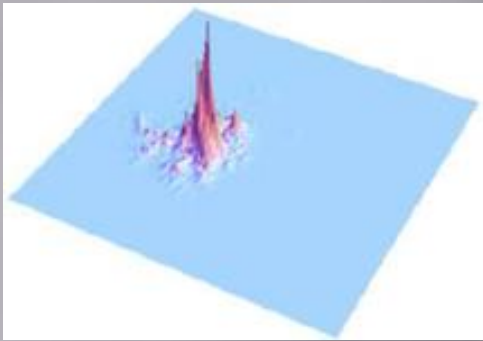
$$\#resonances = \#pairs \times probability = N^2 N^{-\gamma}$$

**Competition of power laws**

$$D = 2 - \gamma$$



# Fractal dimension D and Shannon entropy



**Shannon entropy**

$$-\left\langle \sum_i |\Psi_i|^2 \ln |\Psi_i|^2 \right\rangle = \ln N^* \begin{cases} D = 0, \text{ localized} \\ 0 < D < 1, \text{ fractal} \\ D = 1, \text{ ergodic extended} \end{cases}$$

**N** is size of the system

**D** is the Hausdorff dimension of eigenfunction support set

# Fractal dimension $D(q)$ and Renyi entropy

$$-\ln \left\langle \sum_i |\Psi_i|^{2q} \right\rangle = (q-1)D_q \ln N,$$

$$D_q = \begin{cases} 0, \text{ localized} \\ 0 < D_q < 1, \text{ fractal} \\ 1, \text{ ergodic extended} \end{cases}$$

# Principles of eigenfunction statistics

**Mott criterion:  
sufficient criterion  
of ergodicity**

$$\lim_{N \rightarrow \infty} \frac{\Delta_{hop}}{W} = \infty \quad \Rightarrow \quad \text{weak ergodicity}$$

$W = \sqrt{\langle (H_{mn})^2 \rangle}$ ,  $\Delta_{hop}$  is the spectral band-width of hopping matrix  $\{H_{n \neq m}\}$

**Anderson criterion:  
sufficient criterion  
of localization**

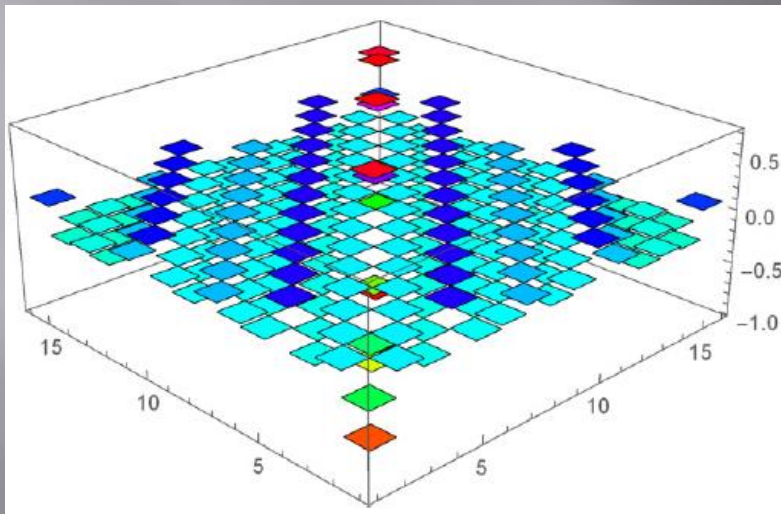
$$\frac{|V_{hop}(R)|}{\delta_R} < R^{-\varepsilon} \quad (\forall \varepsilon > 0), \quad \Rightarrow \quad \text{localization}$$

$\delta_R \sim 1/R$  is mean level spacing in volume  $R$ ,  $V_{hop}(R) = H_{n \neq m}$  is the hopping matrix

$$|V_{hop}(R)| < R^{-(1+\varepsilon)}$$



# Intermediate case: translation-invariant random hopping



$$H_{nm} = V_{n-m}$$

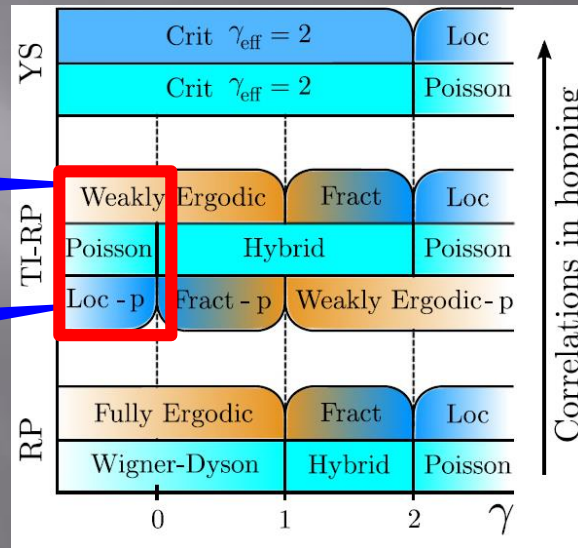
$$\langle |V_{n-m}|^2 \rangle = N^{-\gamma}$$

**Momentum basis is complementary to a coordinate one**

# Coordinate-momentum basis duality and level statistics

Coordinate basis

Momentum basis



$$\gamma_p = 2 - \gamma$$

Poisson level statistics for delocalized (in coordinate space) states

- If the states are localized either in the coordinate or in the momentum basis then the level statistics is Poisson
- If the states are ergodic in any frame then the level statistics is Wigner-Dyson
- Otherwise the level statistics is “hybrid”