A topological criterion: the cases of the AKLT & Kitaev chain

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Tbilisi– 03/06/19
The place…
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… and the team!
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What is a topological phase?

So far, any phase not understood by the Ginzburg-Landau paradigm (i.e. by the breaking, spontaneous or not of a symmetry).

2 main types:

Topological Order ($D \geq 2$)

Symmetry Protected Topological phases (all $D$) = SPT

+ diverse hybrids…
How is a topological phase?

- Its ground state(s) cannot be locally transformed into a product state. For topological orders, the entanglement is always long range, for SPTs, the entanglement may be short-range only. (i.e. non trivial entanglement entropy is non zero)

- The GS degeneracy depends of the topology of the system (i.e. in 1D, the boundary conditions)

- These GS display robust edge states. For SPTs, this robustness against local perturbation is valid only if said perturbation doesn't break the protecting symmetry.

- They display a bulk-edge correspondence (i.e. non zero quantized value of a topological invariant that generates the edge modes)

- They are not differentiable by a local order parameter but a (non-local) topological invariant instead not experimentally accessible.
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How is a topological phase?

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- The GS degeneracy depends of the topology of the system (i.e. in 1D, the boundary conditions)

- These GS display robustness against local perturbation is valid symmetry.

- They display a bulk topological invariant.

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Relevant dynamic areas of research (i.e. shameless advertisement)

Monte Carlo on topological phases

- Use as quantum simulators, for lattice gauge theories in particular

Experimental realisations for all of them

- Understanding of phase transition (other than with CFT)
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Monte Carlo on topological phases

Understanding of phase transition (other than with CFT)

Experimental realisations for all of them

Finding an experimentally relevant unambiguous topological criterion to replace the measurement of the irrelevant local order parameter
Need of a criterion

- Topological Order (D>1)
- Topologically Enriched phases
  - Symmetry-protected topological phases (SPTs)

Disordered and ordered phases through Explicit or Spontaneous Symmetry Breaking (SSB)

Understood by the Ginzburg-Landau paradigm

can mix

can mix
Plan

1. The case of multipartite entanglement entropy.

2. Results on the AKLT chain.

3. Results on the Kitaev wire.
(Non-exhaustive) list of some criteria

- Non-local order parameter
- Edge states
- Entanglement spectrum
- Entanglement entropy
- Fractional statistics
- ...

- Non-generic
- Necessary but not sufficient
- Sufficient but not necessary
- Not experimentally visible with today’s technology
The entanglement entropy(-ies)

- The Von Neumann entanglement entropy:

\[ S_A = - \text{Tr}_A \rho_A \log \rho_A \]

Density matrix (of an isolated system): \( \rho = |\psi\rangle \langle \psi| \)

Reduced density matrix: \( \rho_A = \text{Tr}_B \rho \)

- The Rényi entanglement entropies:

\[ S_A^{(\alpha)} = \frac{1}{1 - \alpha} \log \text{Tr}_A (\rho_A^\alpha) \]

(measurable)
The entanglement entropy(-ies)

- The Von Neumann entanglement entropy:

\[ S_{\text{Von Neumann}}(A:B) = \log \left( \text{Tr} \left( \rho_B \right) \right) - \log \left( \text{Tr} \left( \rho_{AB} \right) \right) \]

- The Rényi entanglement entropies:

\[ S_{\text{Rényi}}(r) = \frac{1}{1 - r} \log \left( \text{Tr} \left( \rho_B^r \right) \right) \]

\[ S_{\text{Rényi}}(r) = \frac{1}{1 - r} \log \left( \text{Tr} \left( \rho_{AB}^r \right) \right) \]

Density matrix (of an isolated system):

\[ \rho_{AB} = \sum_{i} \left| \Phi_i \right\rangle \left\langle \Phi_i \right| \]

Reduced density matrix:

\[ \rho_B = \text{Tr}_A \left( \rho_{AB} \right) \]

The Von Neumann entanglement entropy (bipartite):

Non zero if and only if the density matrix is not a product state between A and B

(if initially a pure state)
An attempt: Wen’s criteria, the mutual information


| Order of the quantum system | Nonzero $I(A:C|B)$ | Zero $I(A:C|B)$ |
|-----------------------------|-------------------|----------------|
| Trivial Order               | $S_{\text{topo}}$, $S_{\text{topo}}^{\text{d}}$ | $S_{\text{topo}}$, $S_{\text{topo}}^{\text{t}}$, $S_{\text{topo}}^{\text{q}}$ |
| Topological Order           | $S_{\text{topo}}^{\text{t}}$, $S_{\text{topo}}^{\text{d}}$ | $S_{\text{topo}}^{\text{t}}$ |
| Symmetry-Breaking Order     | $S_{\text{topo}}^{\text{t}}$ | $S_{\text{topo}}$, $S_{\text{topo}}^{\text{d}}$ |
| Symmetry-Protected Topological Order | $S_{\text{topo}}^{\text{t}}$, $S_{\text{topo}}^{\text{d}}$ | $S_{\text{topo}}$ |

S depends on the cut

Only results of simulation: lack background analytical results (except $S_{\text{topo}}$)
Not easily experimentally accessible (generically)
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The topological entanglement entropy in 1D

\[ S_{\text{topo}}^t = S_{AB} + S_{BC} - S_B - S_{ABC} \]

In the limit where A, B, C and D are big

\[ S_{\text{topo}}^q = S_{AB} + S_{BC} - S_B - S_{ABC} \]
The topological entanglement entropy in 1D

Why ?
(analytically)

\[ S_{\text{topo}} = S_{AB} + S_{BC} - S_B - S_{ABC} \]
Also: measurable?

I. Do protocol to get $s_\alpha$

II. Repeat protocol with same $U_A$ to get $P(s_\alpha)|_{U_A}$

III. Repeat I. with different representative $U_A$ to get $\langle P(s_\alpha) \rangle_{U_A}$ and $\langle P(s_\alpha)^2 \rangle_{U_A}$

IV. Allow access to $S_A^{(2)}$

$$S_{\text{topo}}^{(2)} = S_{AB}^{(2)} + S_{BC}^{(2)} - S_B^{(2)} - S_{ABC}^{(2)}$$

Spoiler alert:

Also: measurable?

I. Do protocol to get $s_\alpha$

II. Repeat protocol with same $U_A$ to get $P(s_\alpha|U_A)$

III. Repeat I. with different representative $U_A$ to get $\langle P(s_\alpha) \rangle_{U_A}$ and $\langle P(s_\alpha)^2 \rangle_{U_A}$

IV. Allow access to $S^{(2)}_A$

$$S_{\text{topo}}^{(2)} = S^{(2)}_{AB} + S^{(2)}_{BC} - S^{(2)}_B - S^{(2)}_{ABC}$$

Spoiler alert: Yes it does, for 1D gapped systems at least

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Need of analytical results: the AKLT model

1D spin-1 chain known to be a SPT (Haldane phase)

\[ H_{\text{AKLT}} = \sum_{k=1}^{N-1} \left[ \vec{S}_k \cdot \vec{S}_{k+1} + \frac{1}{3} \left( \vec{S}_k \cdot \vec{S}_{k+1} \right)^2 \right] \left( + \pi_{0,1} + \pi_{N,N+1} \right) \]

H. Fan, V. Korepin, and V. Roychowdhury, PRL 93, 227203 (2004)

Know the exact ground state(s) depending on the boundary conditions!

«AKLT+»
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How is a topological phase?

- Its ground state(s) cannot be locally transformed into a pure state. For topological orders, it may be shown that the entanglement is always long range, whereas for SPTs, the entanglement may be short-range only. (i.e. non-trivial entanglement entropy is non-zero)

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Protected by (2 amongst 3):
- time-reversal
- inversion
- dihedral group
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- The GS c boundary

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\[ n_{\text{topo}} = 1 \]
Idea: derive all the explicit reduced density matrices and entanglement entropy

\[ \rho_{AUC} = (-1)^L B e^{-L B / \xi} \rho_{A+C} + \left(1 - (-1)^L B e^{-L B / \xi}\right) \rho_A \otimes \rho_C \]

with \( \xi = \log 3 \), and \( \rho_A \) NOT pure!

Generalizable, with defects, including the regular AKLT.
Additivity of the entanglement entropy

$$\text{Limit } L_{B_i} \rightarrow \infty : \quad S(\rho_{n+1}(AKLT+; A_1|A_2| \ldots |A_{n+1})) = \sum_{i=1}^{N+1} S_{L_i}$$

If $$L_i = L : \quad S_{A_1 \cup A_2 \ldots A_{n+1}} = (n + 1) S_L$$

$$= (n + 1) \times 2 \log 2 \quad \text{when } L \rightarrow \infty$$

i.e. contribution of the 2 cut Bell pairs for each subset
Consequences on the other entanglement entropies

For Rényi entropies ($L_i = L$):

$$S_{A_1 \cup \ldots \cup A_n}^{(\alpha)} = \sum_i S_{A_i}^{(\alpha)}$$

Measurable for $\alpha = 2$

For Wen’s « $q$ » topological entanglement entropy:

$$S_{\text{topo}}^q = S_{AB} + S_{BC} - S_B - S_{ABC} = 2 \log 2$$

For -all- size and position of partition, if big enough
To do with AKLT-like systems

- Check the validity of the criterion for the whole phase diagram (numerically).
  - Done for the bilinear biquadratic model and agreement!

- Check the what happens for generalization of the Haldane phase: the contribution of one « Bell pair » increases, hence getting direct access to the virtual edge states, almost the topological invariant.
  - Main future direction + generalization for all SPTs.

- Measure it experimentally.
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The Kitaev wire with interaction

1D chain of spinless fermions:

$$H = \sum_{j=1}^{L-1} \left( -t \left( a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger \right) + 4U \left( a_j^\dagger a_j - \frac{1}{2} \right) \left( a_{j+1}^\dagger a_{j+1} - \frac{1}{2} \right) - \mu \sum_{j=1}^{L} \left( a_j^\dagger a_j - \frac{1}{2} \right)$$

For $\Delta = t$, symmetric with $\mu \rightarrow -\mu$

Extracted from:

Hosho Katsura, Dirk Schuricht, and Masahiro Takahashi
The topological phase of the Kitaev wire

Periodic BC: GS x 1
Open BC: GS x 2
Non locally transformable into a pure state
2 Majorana edge states

Protected by $\mathbb{Z}_2$ topological invariant.
The results found for the AKLT are the same for the Kitaev topological phase.

Separability of the multipartite reduced density matrix (into MIXED states, $\xi = 0$ !):

$$\rho_{A_1 \cup \ldots \cup A_{n+1}} = \rho_{A_1} \otimes \ldots \otimes \rho_{A_{n+1}}$$

Additivity:
(for the Von Neumann)

$$S_{A_1 \cup A_2 \ldots \cup A_{n+1}} = (n+1) \times \log 2$$

Bell pair per 2 cuts.

Additivity:
(for the Rényi)

$$S^{(\alpha)}_{A_1 \cup A_2 \ldots \cup A_n} = \sum_i S^{(\alpha)}_{A_i}$$

Non nullity of Wen's criteria:

$$S^q_{\text{topo}} = S_{AB} + S_{BC} - S_B - S_{ABC} = \log 2$$
The phase diagram in terms of the criterion
Scalability

$\mu = 1$
$\Delta = t = 1$

N=50
U=0

$\Delta = t = 1$

Obtained by free fermion technique

Obtained by DMRG

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03/06/19
Entanglement critical exponent

Curve intersection

\[ S_{\text{topo}}^q L^{a} = \lambda \left( L^{b} (\alpha - \alpha_c) \right) \]

Get \( \alpha_c \)

Get \( a = b = 1 \)
Conclusion

- Topological characteristic recognizable by entanglement, in particular in 1D using $S_{\text{topo}}^q$.
- Can use the Rényi entropies just as easily, that are measurable.
- Interpretation in terms of static regular Bell pairs along the chain.
- $S_{\text{topo}}^q$ may be used like an order parameter for many things.
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Thank you for your attention !
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