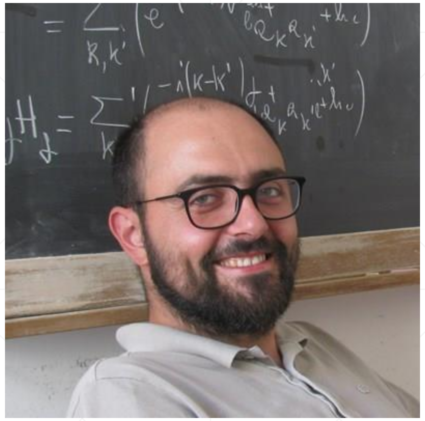
A topological criterion : the cases of the AKLT & Kitaev chain

By Pierre Fromholz –
With M. Dalmonte and G. Magnifico
Tbilisi– 03/06/19

The place…

ICTP, Trieste, Italy

… and the team !



Marcello Dalmonte

Giuseppe Magnifico

What is a topological phase ?

So far, any phase not understood by the Ginzburg-Landau paradigm (i.e. by the breaking, spontaneous or not of a symmetry).

2 main types :

Topological Order ($D \geq 2$)

Symmetry Protected Topological phases (all D)
=SPT

+ diverse hybrids…

How is a topological phase ?

- Its ground state(s) cannot be locally transformed into a product state. For topological orders, the entanglement is always long range, for SPTs, the entanglement may be short-range only. (i.e. non trivial entanglement entropy is non zero)
- The GS degeneracy depends of the topology of the system (i.e. in 1D, the boundary conditions)
- These GS display robust edge states. For SPTs, this robustness against local perturbation is valid only if said perturbation doesn't break the protecting symmetry.
- They display a bulk-edge correspondence (i.e. non zero quantized value of a topological invariant that generates the edge modes)
- They are not differentiable by a local order parameter but a (non-local) topological invariant instead not experimentally accessible.

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$$\nexists U \ \text{local} \ U|\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_L\rangle$$

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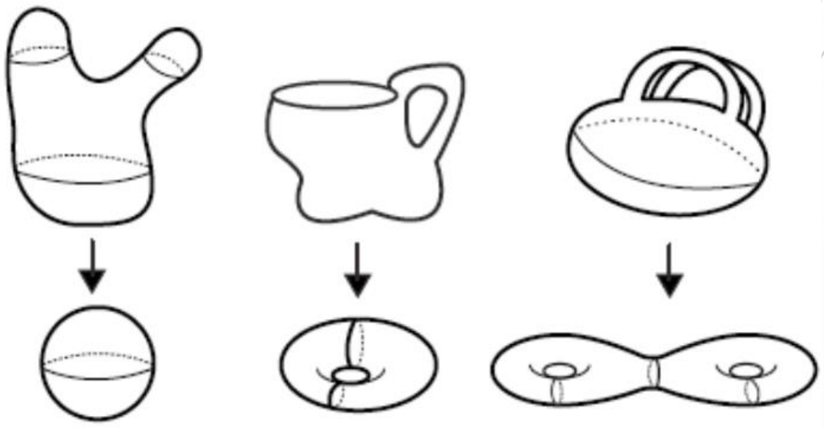

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How is a topological phase ?

Bulk ⟷ Topological invariant ⟷ Topological defect ⟷ Edge

- Its ground state(s) cannot be locally transformed into a pure state. For topological orders, the entanglem … or SPTs, the entanglement may be short-range only. (i … entropy is non zero)

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- The GS degene… in 1D, the boundary condi…

$$\mathcal{O}^{\alpha}_{\mathrm{str}} = -\lim_{|j-i|\to\infty} \left\langle S_i^{\alpha} \exp\left(i\pi \sum_{i<k<j} S_k^{\alpha} \right) S_j^{\alpha} \right\rangle$$

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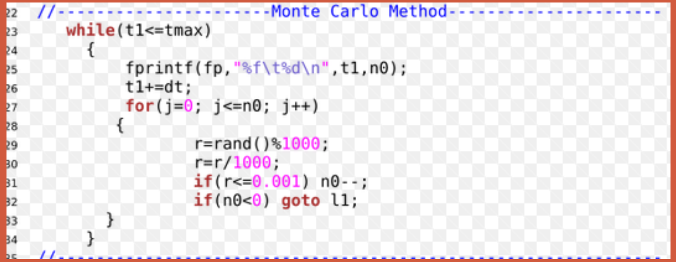
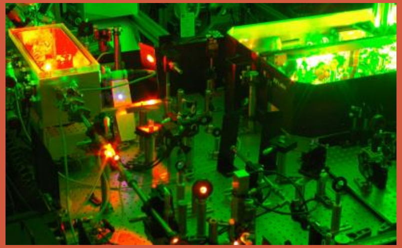
Relevant dynamic areas of research (i.e. shameless advertisement)

Monte Carlo on topological phases

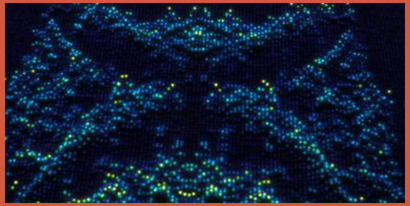


```
//---------------------Monte Carlo Method---------------------
  while(t1<=tmax)
    {
        fprintf(fp,"%f\t%d\n",t1,n0);
        t1+=dt;
        for(j=0; j<=n0; j++)
       {
              r=rand()%1000;
              r=r/1000;
              if(r<=0.001) n0--;
              if(n0<0) goto l1;
       }
    }
```

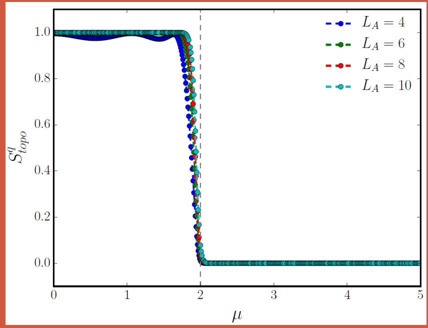
Experimental realisations for all of them



Use as quantum simulators,
for lattice gauge theories in particular



Understanding of phase transition
(other than with CFT)



Relevant dynamic areas of research (i.e. shameless advertisement)

Monte Carlo on topological phases

```
//----
whi
{
}
//----
```

Experimental realisations for all of them

Finding an experimentally relevant unambiguous topological criterion to replace the measurement of the irrelevant local order parameter

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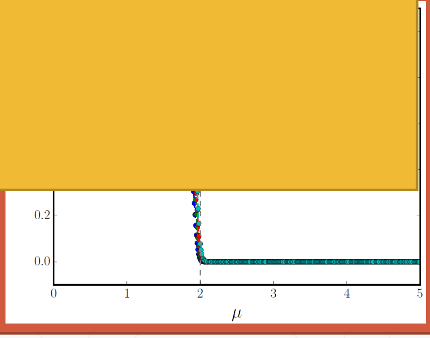


Need of a criterion

Topological Order (D>1)

Topologically Enriched phases

can mix

Symmetry-protected topological phases (SPTs)

Disordered and ordered phases through Explicit or Spontaneous Symmetry Breaking (SSB)

can mix

Understood by the Ginzburg-Landau paradigm

Plan

1. The case of multipartite entanglement entropy.
2. Results on the AKLT chain.
3. Results on the Kitaev wire.

(Non-exhaustive) list of some criteria

- Non-local order parameter
- Edge states
- Entanglement spectrum
- Entanglement entropy
- Fractional statistics
- …

Non-generic

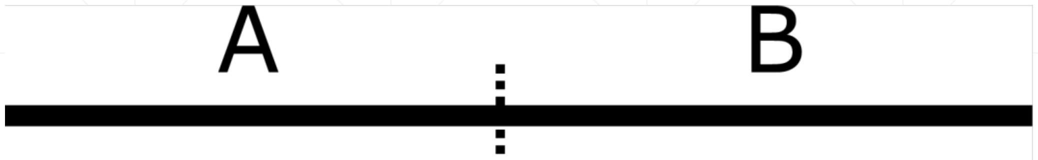
Necessary but not sufficient

Sufficient but not necessary

Not experimentally visible with today's technology

The entanglement entropy(-ies)

- The Von Neumann entanglement entropy :



Density matrix (of an isolated system) : $\rho = |\psi\rangle\langle\psi|$

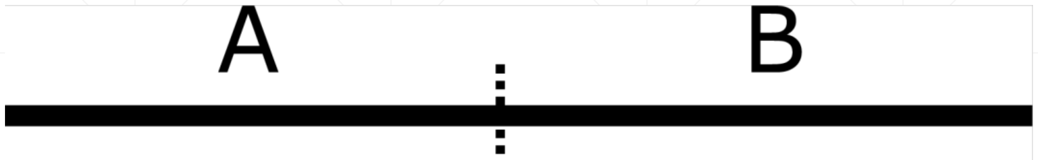
Reduced density matrix : $\rho_A = \mathrm{Tr}_B\, \rho$

The Von Neumann entanglement entropy (bipartite) : $S_A = -\,\mathrm{Tr}_A\, \rho_A \log \rho_A$

- The Rényi entanglement entropies : $S_A^{(\alpha)} = \dfrac{1}{1-\alpha} \log \mathrm{Tr}_A \left(\rho_A^\alpha\right)$
  (measurable)

The entanglement entropy(-ies)

- The Von Neumann entanglement entropy :



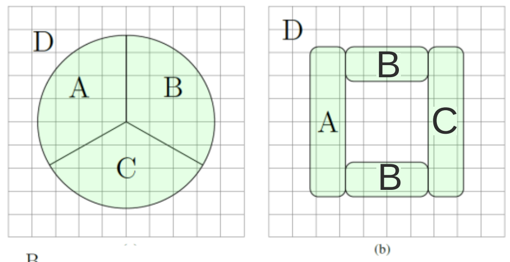
Non zero if and only if the density matrix is not a product state between A and B
(if initially a pure state)

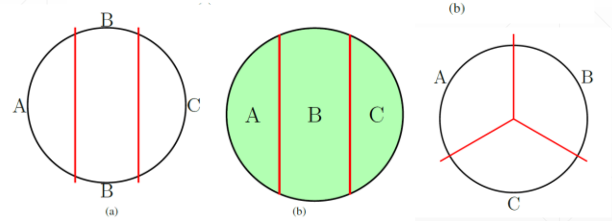
An attempt : Wen’s criteria, the mutual information

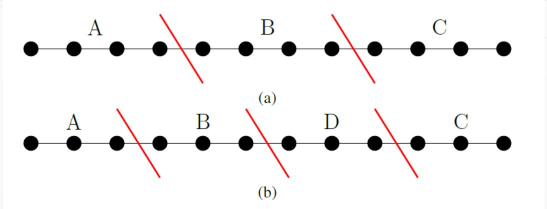
B. Zeng, X. Chen, D.-L. Zhou, X.-G. Wen arXiv:1508.02595v4

Order of the quantum system	Nonzero $I(A{:}C\|B)$	Zero $I(A{:}C\|B)$
Trivial Order		$S_{\text{topo}}, S^{\text{t}}_{\text{topo}}, S^{\text{q}}_{\text{topo}}$
Topological Order	$S_{\text{topo}}, S^{\text{q}}_{\text{topo}}$	$S^{\text{t}}_{\text{topo}}$
Symmetry-Breaking Order	$S^{\text{t}}_{\text{topo}}$	$S_{\text{topo}}, S^{\text{q}}_{\text{topo}}$
Symmetry-Protected Topological Order	$S^{\text{t}}_{\text{topo}}, S^{\text{q}}_{\text{topo}}$	S_{topo}

S depends on the cut







Only results of simulation : lack background analytical results (except S_{topo})
Not easily experimentally accessible (generically)

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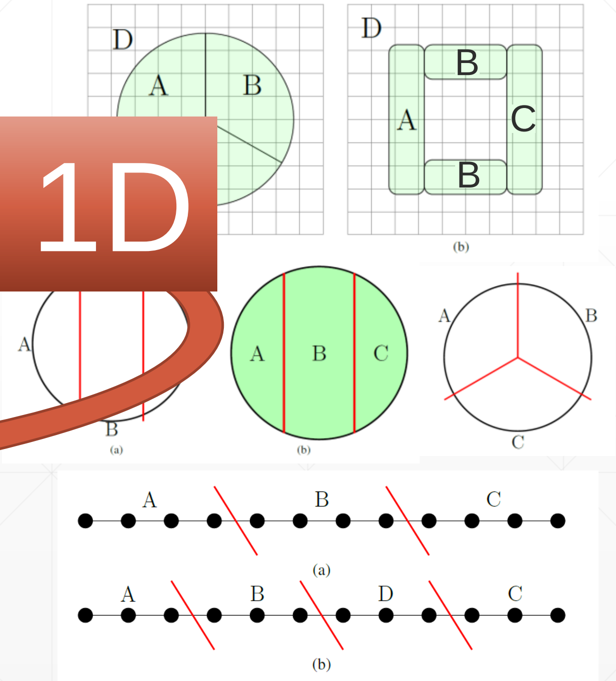
In 1D

S depends on the cut



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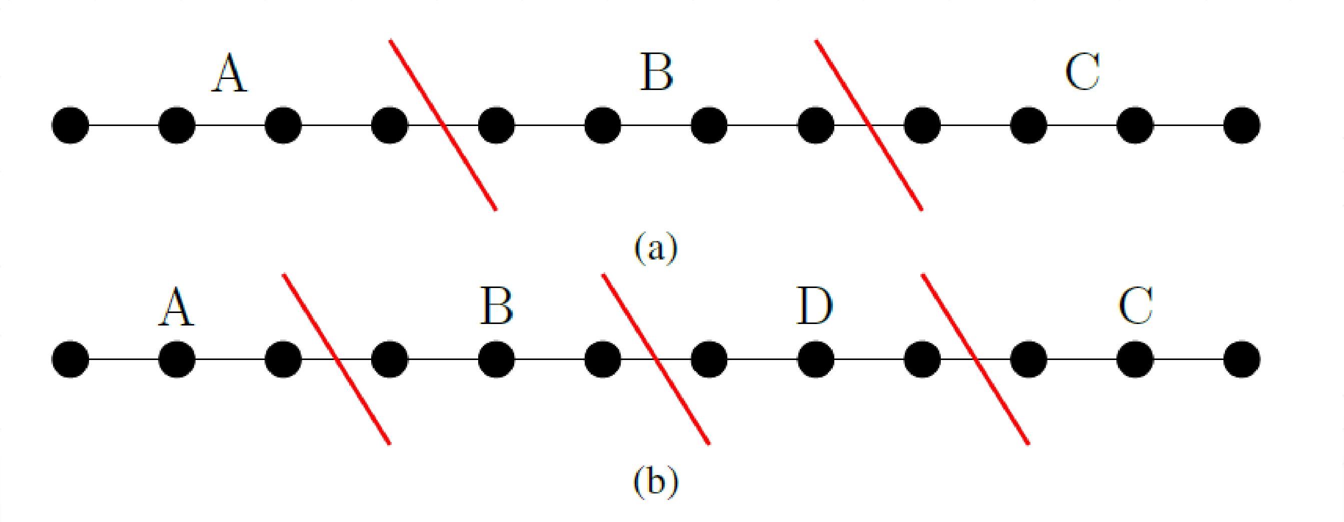
The topological entanglement entropy in 1D

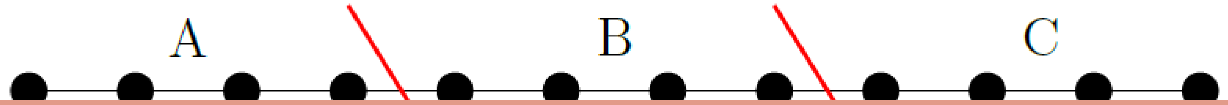


a) $$S^{t}_{\text{topo}} = S_{AB} + S_{BC} - S_{B} - S_{ABC}$$

b) $$S^{q}_{\text{topo}} = S_{AB} + S_{BC} - S_{B} - S_{ABC}$$

In the limit where A,B,C and D are big

The topological entanglement entropy in 1D



Why ?
(analytically)

a

ere
are big

b) $S^q_{\text{topo}} = S_{AB} + S_{BC} - S_B - S_{ABC}$

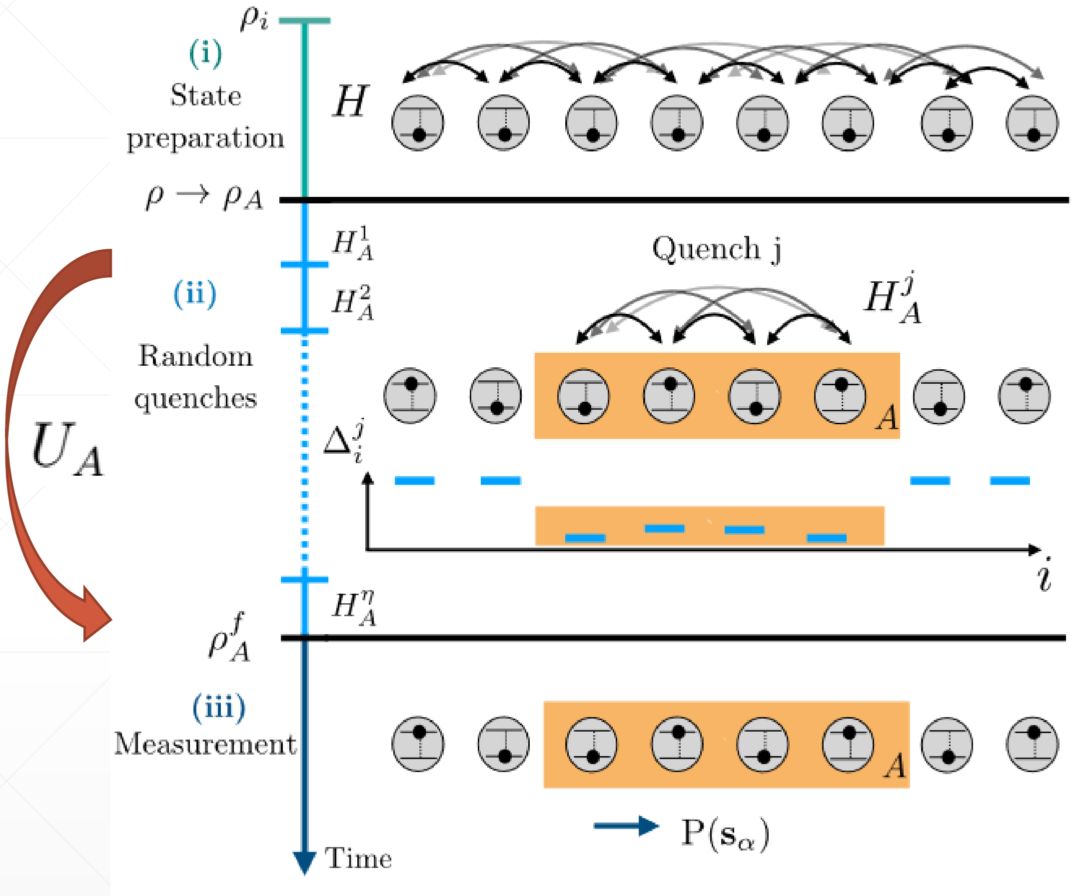
Also : measurable ?

I. Do protocol to get $\mathbf{s}_\alpha$

II. Repeat protocol with same U_A to get $P(\mathbf{s}_\alpha)|_{U_A}$

III. Repeat I. with different representative U_A to get $\langle P(\mathbf{s}_\alpha)\rangle_{U_A}$ and $\langle P(\mathbf{s}_\alpha)^2\rangle_{U_A}$

IV. Allow access to $S_A^{(2)}$



$$S_{\text{topo}}^{q,(2)} = S_{AB}^{(2)} + S_{BC}^{(2)} - S_{B}^{(2)} - S_{ABC}^{(2)} \quad ?$$

B. Vermersch, A. Elben, M. Dalmonte, J. I. Cirac, and P. Zoller, Phys. Rev. A 97, 023604 (2018)

Spoiler alert :

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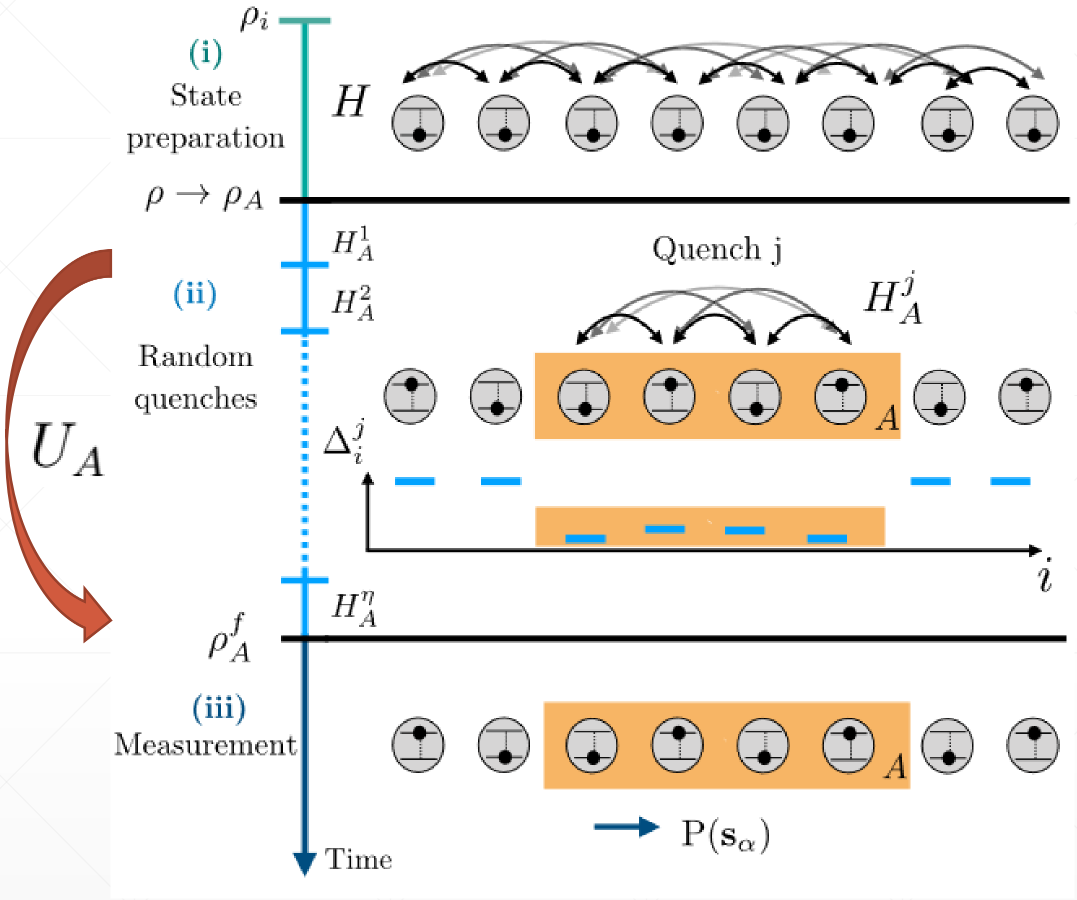

B. Vermersch, A. Elben, M. Dalmonte, J. I. Cirac, and P. Zoller, Phys. Rev. A 97, 023604 (2018)

Spoiler alert : Yes it does, for 1D gapped systems at least

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Need of analytical results : the AKLT model

1D spin-1 chain known to be a SPT (Haldane phase)

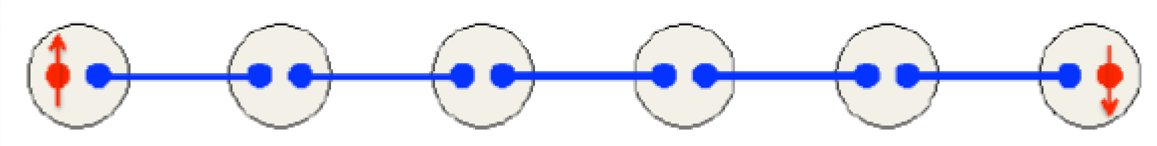
A. Affleck, T. Kennedy, E.H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987)

$$H_{\mathrm{AKLT}} = \sum_{k=1}^{N-1} \left[\vec{S}_k \vec{S}_{k+1} + \frac{1}{3} \left(\vec{S}_k \vec{S}_{k+1} \right)^2 \right] \quad (+\pi_{0,1} + \pi_{N,N+1})$$

H. Fan, V. Korepin, and V. Roychowdhury, PRL **93**, 227203 (2004)

« AKLT+ »

Know the exact ground state(s) depending on the boundary conditions !



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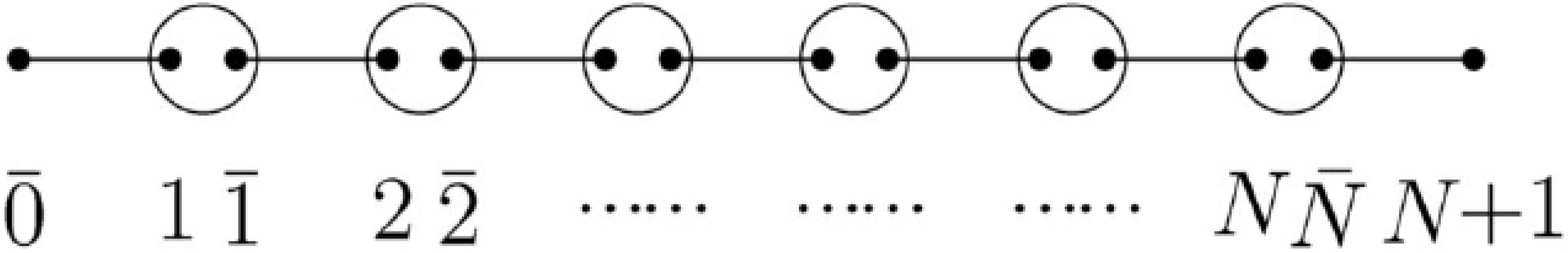

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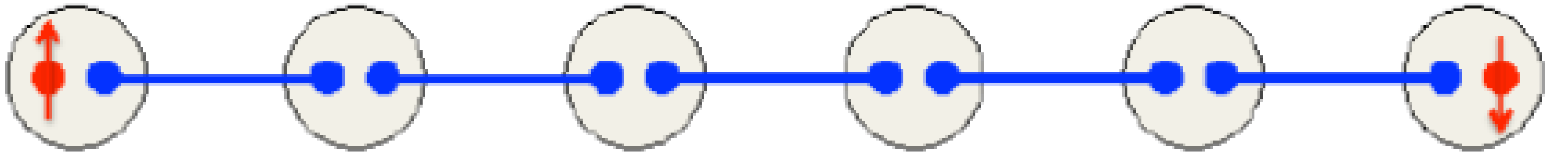
How is a topological phase ?

- Its g… orde… be s… polo… ent m…



x4

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x1

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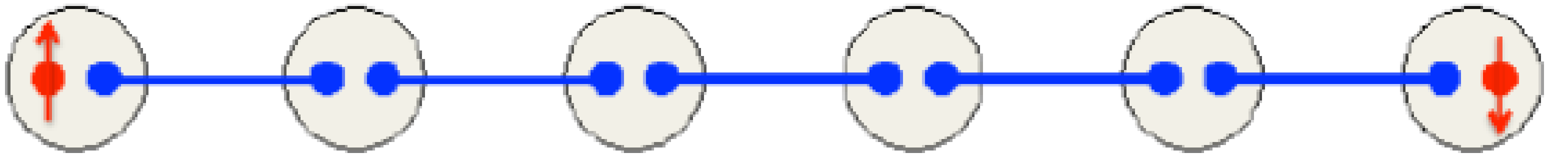

- The G… boundary conditions)
- These GS display robust edge states. For SPTs, this robustness against local perturbation is valid only if said perturbation doesn't break the protecting symmetry.

Protected by (2 amongst 3) :
-time-reversal
-inversion
-dihedral group

…of a

…ological invariant instead not experimentally accessible.

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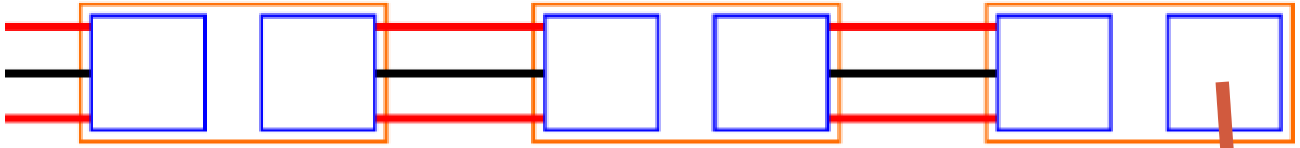

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n(topo)=1

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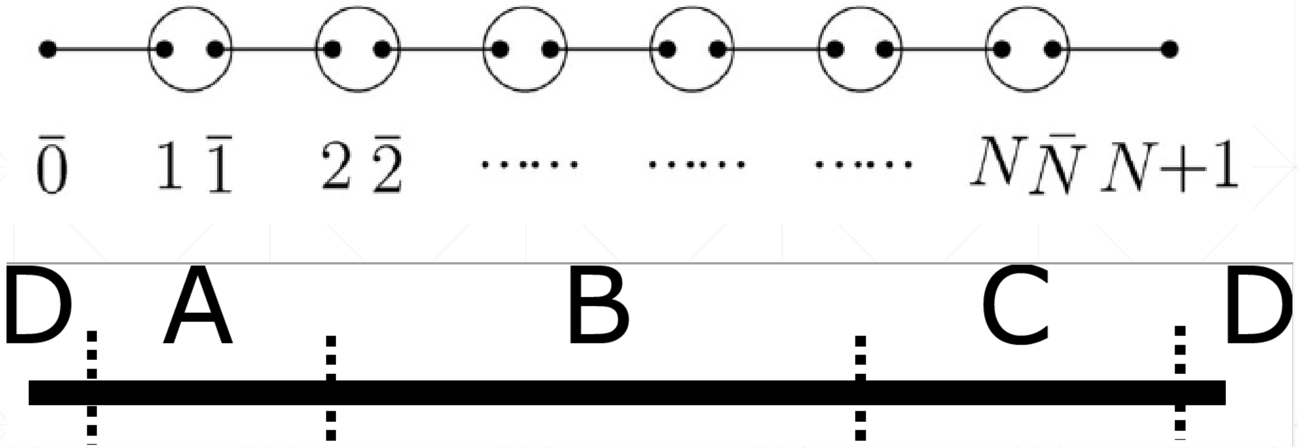
Idea : derive all the explicit reduced density matrices and entanglement entropy

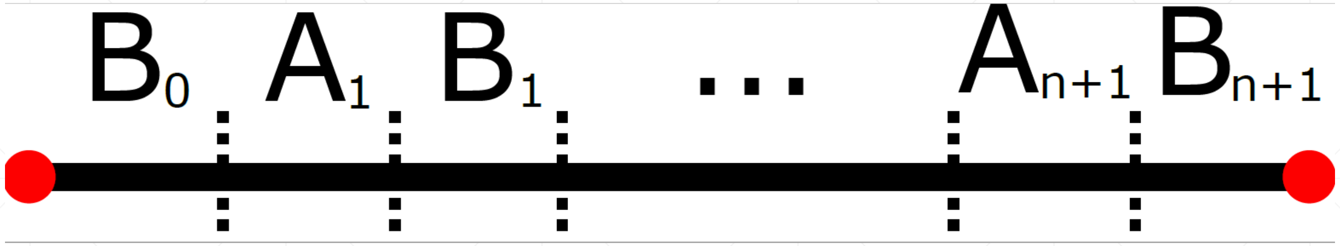


AKLT+

$\rho_{A\cup C} = (-1)^{L_B} e^{-L_B/\xi} \rho_{A+C} + \left(1 - (-1)^{L_B} e^{-L_B/\xi}\right) \rho_A \otimes \rho_C$ with $\xi = \log 3$, and ρ_A NOT pure !

Generalizable, with defects, including the regular AKLT.

Additivity of the entanglement entropy

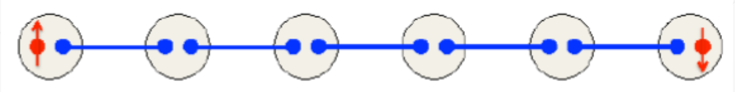


Limit $L_{B_i} \to \infty$: $S(\rho_{n+1}(AKLT+;A_1|A_2|\ldots|A_{n+1})) = \sum_{i=1}^{N+1} S_{L_i}$

If $L_i = L$: $S_{A_1 \cup A_2 \ldots A_{n+1}} = (n+1)\, S_L$

$= (n+1) \times 2 \log 2$ when $L \to \infty$

i.e. contribution of the 2 cut Bell pairs for each subset



Consequences on the other entanglement entropies

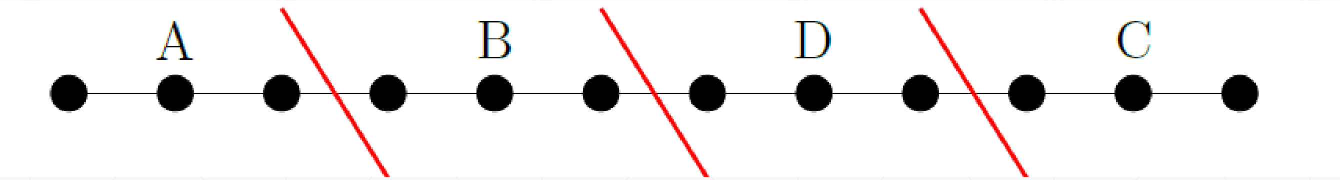
For Rényi entropies ($L_i = L$) : $S^{(\alpha)}_{A_1 \cup \ldots A_n} = \sum_i S^{(\alpha)}_{A_i}$

Measurable for $\alpha = 2$

For Wen's « q » topological entanglement entropy :

$$S^q_{\text{topo}} = S_{AB} + S_{BC} - S_B - S_{ABC} = 2 \log 2$$

For -all- size and position of partition, if big enough



To do with AKLT-like systems

- Check the validity of the criterion for the whole phase diagram (numerically).

  Done for the bilinear biquadratic model and agreement !

- Check the what happens for generalization of the Haldane phase : the contribution of one « Bell pair » increases, hence getting direct access to the virtual edge states, almost the topological invariant.

  Main future direction + generalization for all SPTs.

- Measure it experimentally.

Plan

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The Kitaev wire with interaction

1D chain of spinless fermions :

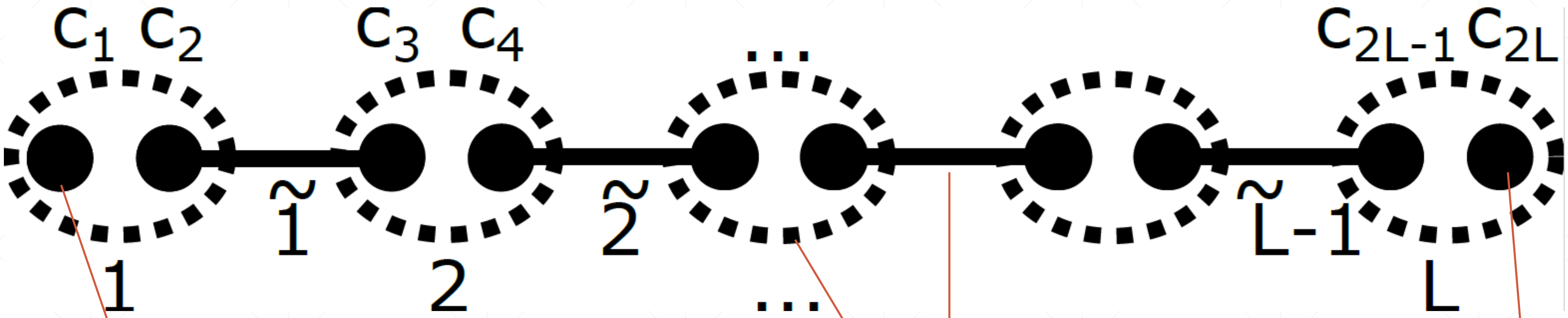
$$H = \sum_{j=1}^{L-1} \left(-t \left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right) + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger \right.$$
$$\left. + 4U \left(a_j^\dagger a_j - \frac{1}{2} \right) \left(a_{j+1}^\dagger a_{j+1} - \frac{1}{2} \right) \right) - \mu \sum_{j=1}^{L} \left(a_j^\dagger a_j - \frac{1}{2} \right)$$

For $\Delta = t$, symmetric with $\mu \to -\mu$

Extracted from :

Hosho Katsura, Dirk Schuricht, and Masahiro Takahashi
Phys. Rev. B 92,115137, (2015)

The topological phase of the Kitaev wire



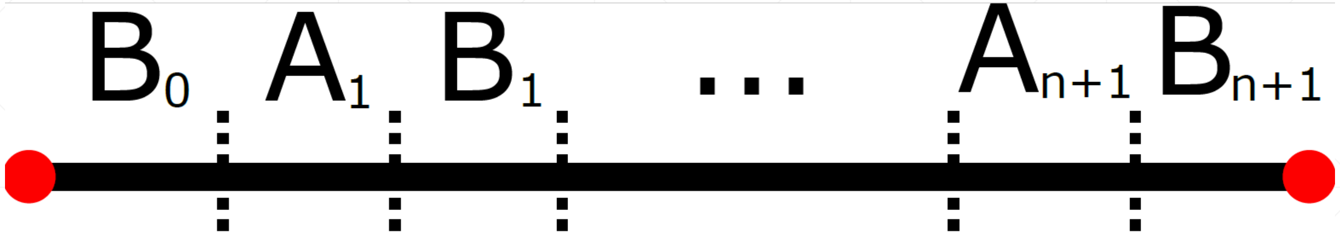
Periodic BC : GS x 1
Open BC : GS x 2

Non locally transformable into a pure state

2 Majorana edge states

Protected by Z_2 topological invariant.

The results found for the AKLT are the same for the Kitaev topological phase.



« Kitaev+»

$\Delta = t = 1$

$U, \mu = 0$

Separability of the multipartite reduced density matrix (into MIXED states, $\xi = 0$!) :

$$\rho_{A_1 \cup \ldots A_{n+1}} = \text{"}\rho_{A_1} \otimes \ldots \rho_{A_{n+1}}\text{"}$$

here, for fermionic states : there can be a sign !

Additivity :
(for the Von Neumann)

$$S_{A_1 \cup A_2 \ldots A_{n+1}} = (n+1) \times \log 2$$

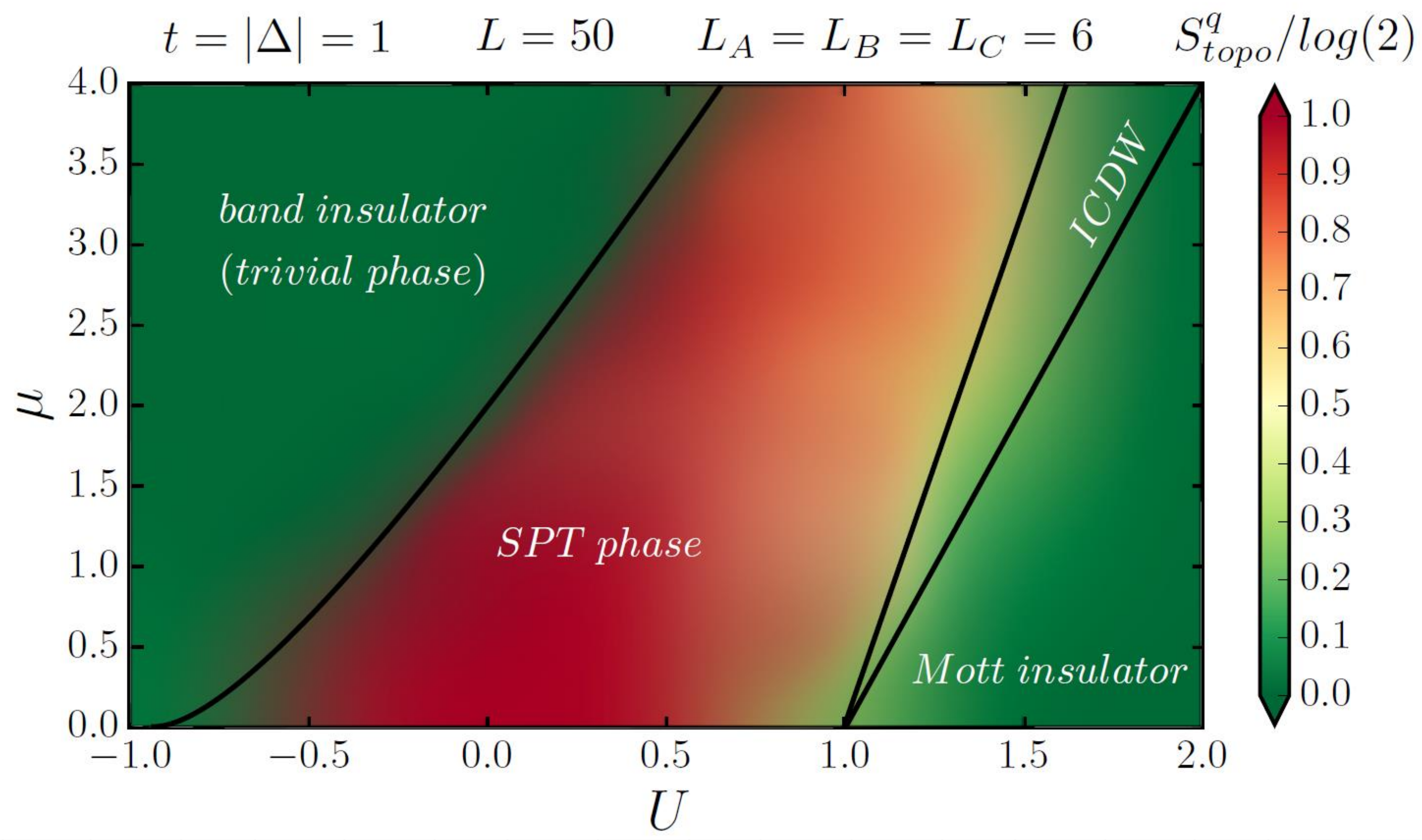
Bell pair per 2 cuts.

Additivity :
(for the Rényi)

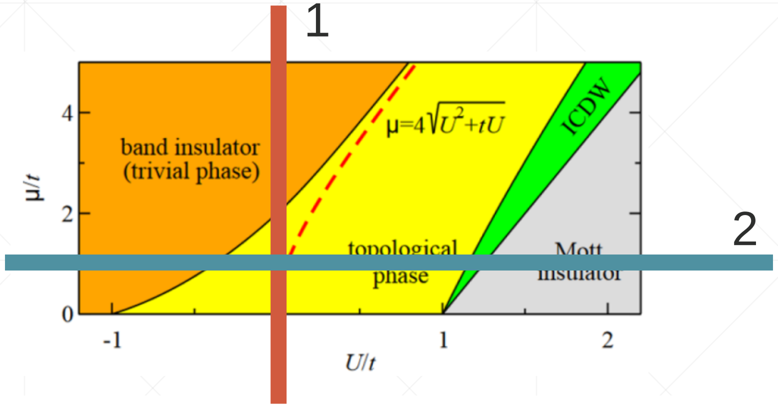
$$S^{(\alpha)}_{A_1 \cup \ldots A_n} = \sum_i S^{(\alpha)}_{A_i}$$

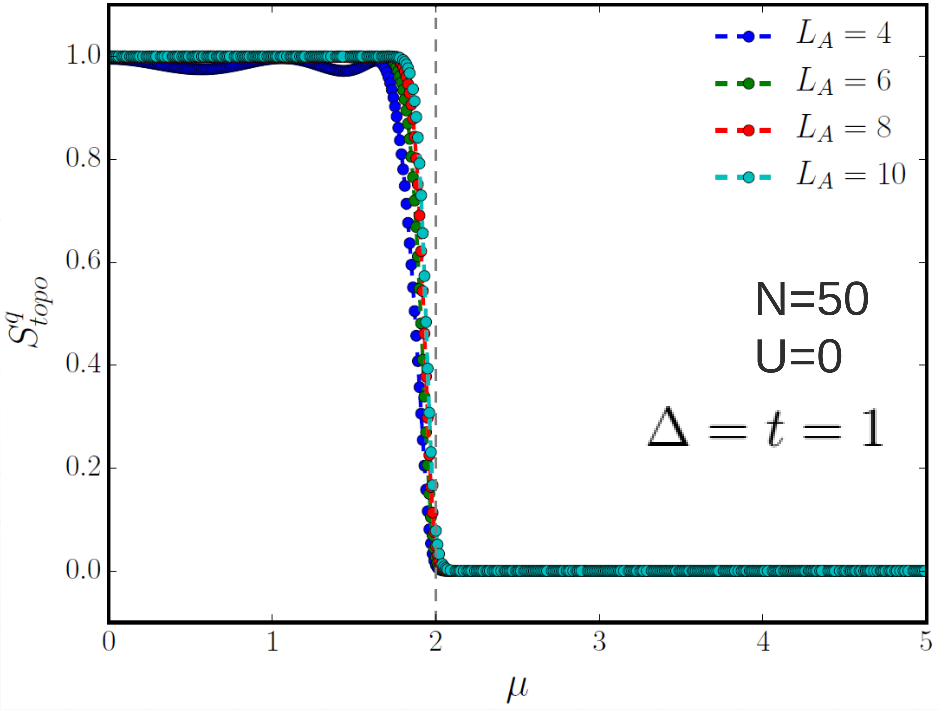
Non nullity of Wen's criteria : $S^q_{\text{topo}} = S_{AB} + S_{BC} - S_B - S_{ABC} = \log 2$

The phase diagram in terms of the criterion

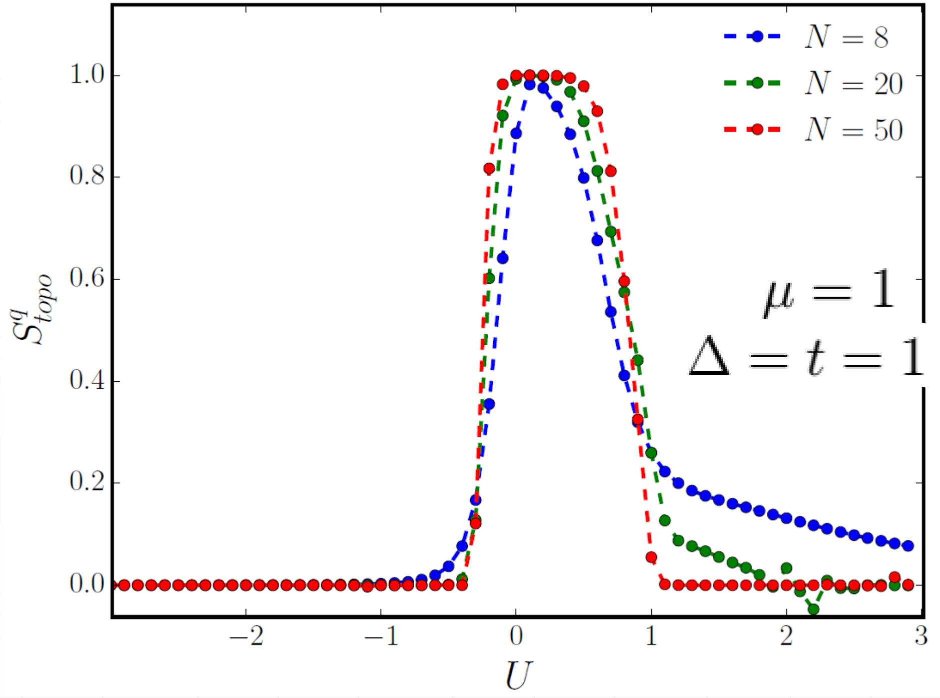


Scalability

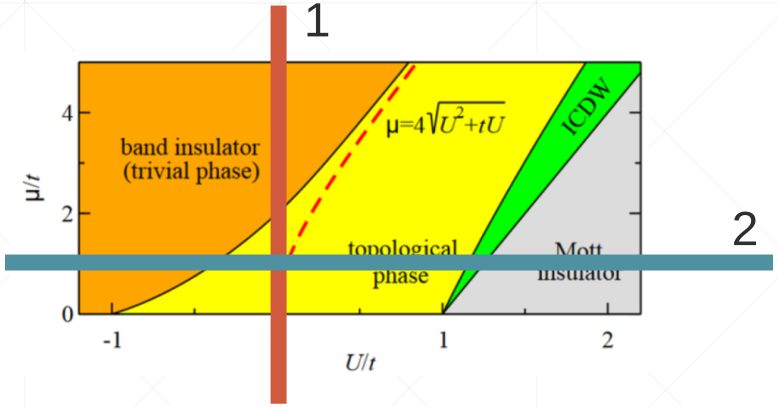




Obtained by free fermion technique



Obtained by DMRG



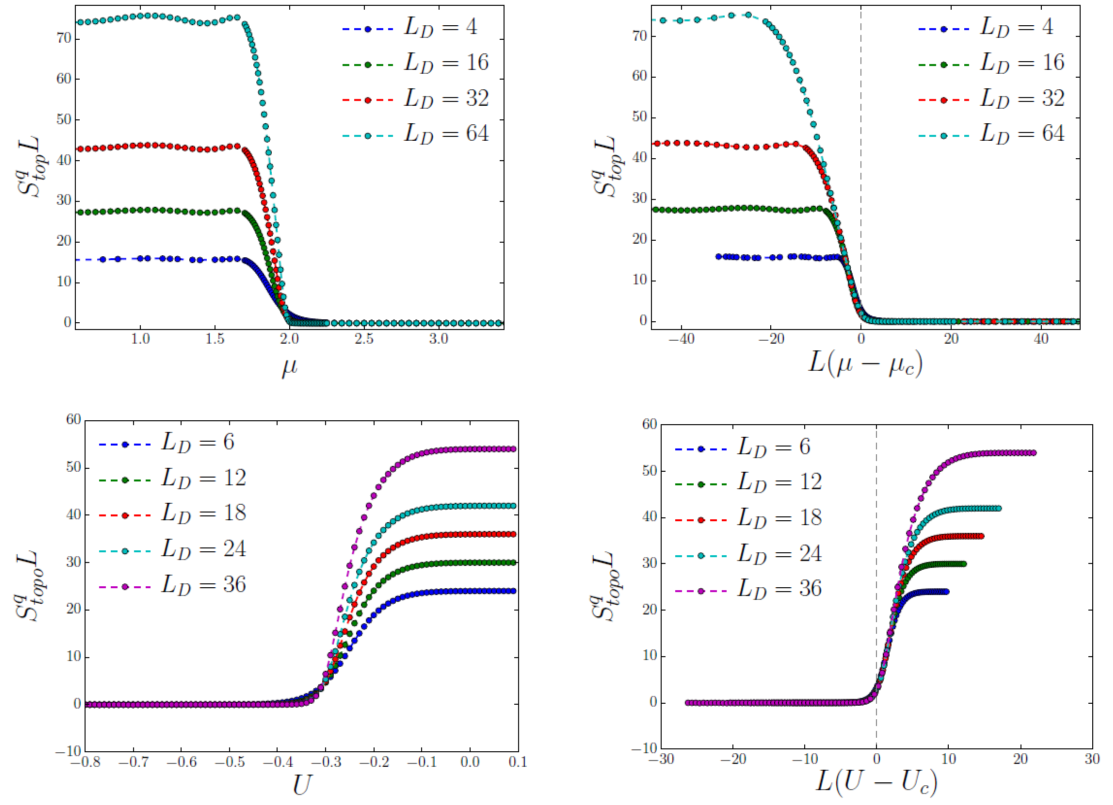
Entanglement critical exponent

Curve intersection

Curve collapse

1

2
(left)



$$S^q_{topo} L^{\frac{a}{b}} = \lambda\left(L^{\frac{1}{b}}(\alpha - \alpha_c)\right)$$

Get α_c

Get $a = b = 1$!

Conclusion

- Topological characteristic recognizable by entanglement,
  in particular in 1D using S^q_{topo}.
- Can use the Rényi entropies just as easily, that are measurable.
- Interpretation in terms of static regular Bell pairs along the chain.
- S^q_{topo} may be used like an order parameter for many things

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Thank you for your attention !

Plan

1. The case of multipartite entanglement entropy.
2. Results on the AKLT chain.
3. Results on the Kitaev wire.